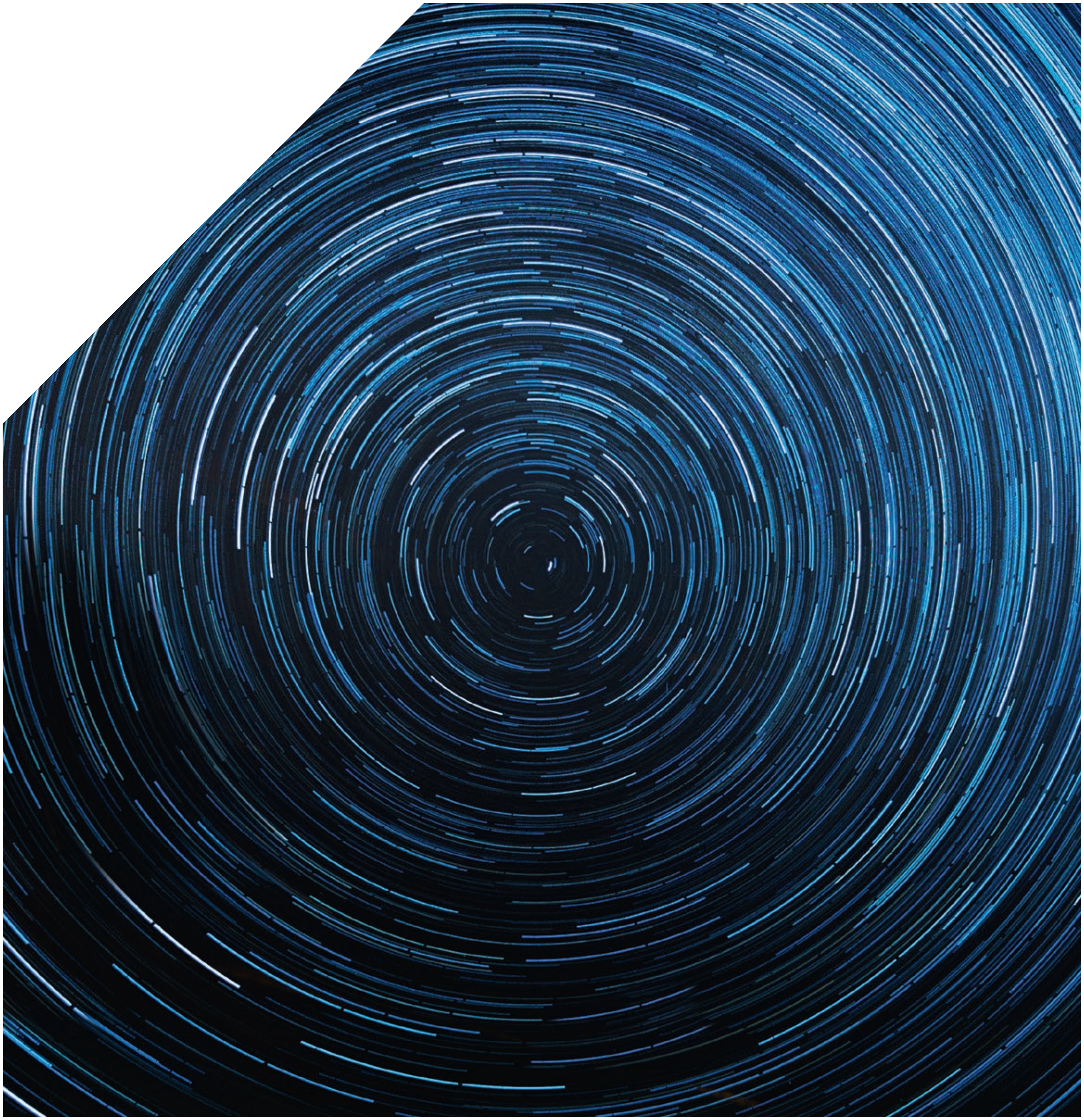
# Invesco Vision

Portfolio management decision support system



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# Contents

# Foreword

The exercise of translating a specified future investment objective into a strategy to be implemented in the present requires vision. It necessitates the ability to effectively evaluate today's opportunities, conceptualize the risks that may materialize across an investment horizon, and implement a plan to manage those risks over time. It is a process that requires ongoing introspection about the preferences for the different risks that come to light as markets evolve and the judgment that must be applied when uncertainty prevails.

Invesco's long history in managing investments and facing the real-world challenges of allocating capital across dynamic and volatile markets has afforded us the understanding that investing is about making decisions about the trade-offs between the possibility of higher returns and the risks that accompany the pursuit of those returns. Our experience has also taught us that those trade-offs aren't always evident. Making the right decisions requires consideration of the right information. It is this aspect of investing that led the Invesco Investment Solutions (IIS) group to pursue the development of differentiated capabilities focused on supporting the investment decision making process. Invesco Vision is the result of our ongoing pursuit of this idea.

IIS's team of global research professionals, with expertise across a variety of domains (e.g., Mathematics, Statistics, Data Science, etc.), has dedicated years of research and effort to developing the Invesco Vision portfolio management decision support system. The platform was specifically designed around the idea of providing professional investors with the information they need to make better informed investing decisions. We created Invesco Vision to allow for more productive collaborations with our clients and to support them in most effectively applying their judgment in the portfolios they manage.

We are excited to share an overview of Invesco Vision's current capabilities with you. However, we are steadfast in our commitment to this effort and look forward to advancing Vision as technology, markets, and investors evolve.

**Gary Wendler**
Head of Investment Solutions,
Product Strategy and Investment
Performance & Risk

**Nicholas Savoulides, PhD, CFA**
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# Introduction

*When every individual person labours apart, and only for himself, his force is too small to execute any considerable work; his labour being employ'd in supplying all his different necessities, he never attains a perfection in any art; and as his force and success are not at all times equal, the least failure in either of these particulars must be attended with inevitable ruin and misery. Society provides a remedy for these three inconveniences. By the conjunction of forces, our power is augmented: By the partition of employments, our ability encreases: And by mutual succour we are less expos'd to fortune and accidents. 'Tis by this additional force, ability, and security, that society becomes advantageous.*

David Hume, A Treatise of Human Nature (1739)

Investment management is about seeking returns in the face of uncertainty. While we cannot control uncertainty, we can affect the decisions we make in managing investments. In this context, the role of a portfolio manager is to make judgments, informed by experience and relevant information, about investments being considered and the uncertainty they present. As a stand-alone exercise, this is no small task. However, portfolio managers are increasingly required to be "expert generalists" about an expanding array of ideas that are now a part of investment management. Managing portfolios entails navigating dynamic and politically charged macro-economic environments, evaluating emerging technologies and new approaches to investing, understanding and applying advanced quantitative methods, complying with changing regulatory requirements across multiple regions, and executing the effective implementation of investment strategy. Arguably, the knowledge and skills now demanded of portfolio managers spans a broad array of disciplines that extend well beyond the domain of traditional finance. Unfortunately, the informational challenges confronting investors today are not likely to subside as the financial markets continue to evolve at an ever-increasing pace. Consequently, having access to trusted resources for collaboration that can facilitate access to relevant information and analytics that supports investing decision making is becoming a critical aspect of successful asset management. This is why we created Invesco Vision.

Invesco Vision is a portfolio management decision support system that provides Invesco's researchers and clients with a broad set of capabilities that allow for the development of insights about the risks and trade-offs presented by individual assets and portfolios to support the identification of solutions that are best aligned with an investor's specific preferences. With this system we seek to facilitate the application of judgment to investment decisions through collaboration and the effective division of labor between human and machine; where judgment (about objectives, approaches/methods to use, and about a wider range of considerations than can ever be incorporated into a computer) is left to the investor, while computation (of sophisticated algorithms, methods for portfolio optimization, and estimates of potential risks and rewards) is left to the machine. Invesco Vision is supported by a seasoned group of Investment Solutions Strategists with practical experience and a team of global research professionals tasked with the evaluation and development of the methodologies and capabilities that are made available through the system.

The objective of this paper is to provide investors with the necessary detail about the approaches and methods incorporated into Invesco Vision. This allows for a common understanding that can serve as the basis for effective engagements with our clients in addressing their investment challenges. To that end, we have divided the paper into four parts that each provide concise descriptions of the methods and techniques used in the system and how they might be applied in practice. The first part, Modeling assets and liabilities, provides detail on the development of various covariance matrices and how they are used to inform the requisite inputs for risk evaluation and portfolio construction. The second part, Portfolio construction, provides information about the various portfolio optimization methods available. The third part, Portfolio analytics, explains some of the key analytical tools that can be used for both portfolio evaluation and portfolio selection. The fourth and final part, Practical application: Case studies, presents fifteen different case studies intended to provide simplified examples of how Invesco Vision can be used to address common investment challenges.

By providing information and transparency about how we approach portfolio construction and risk management, we hope that this paper can serve as a constructive first step toward establishing client partnerships where investing experiences and outcomes are meaningfully improved through collaboration.

# Modeling assets and liabilities

**Covariance estimation**

Successful investing requires an understanding of asset and market dynamics. For portfolio management, the covariance matrix is essentially a model of "how the world works" in that it describes both the volatilities of, and the relationships between, investments under consideration. This information is central to the development of efficient portfolios and for risk management exercises. Unfortunately, developing a model of covariance that quantifies how assets might be expected to act in the future is not a simple task. Invesco has expended considerable effort to identify a framework that provides relevant information for portfolio construction and risk assessment. Our focus was on identifying a multi-factor risk model that provided a high degree of flexibility and allowed for the consistent modeling of a broad range of assets.

Multi-factor risk models generally fall into three main categories: 1) macroeconomic models, 2) statistical models, and 3) fundamental models. Macroeconomic models are perhaps the most simple and intuitive in that they use observable economic time series (e.g., GDP, interest rates, inflation, credit spreads, etc.) to explain the returns of, and relationships between, assets. Conversely, statistical factor models are substantially less intuitive because they rely on unobservable statistical factors, generally derived from a factor analysis or principal components analysis, for explanatory purposes. Determining sensitivities, or betas, to either macroeconomic or statistical factors is accomplished through time series regression. This is a key limitation for these types of models in that the statistically sound estimation of exposures requires long return histories. In many instances, sufficiently long historical information is not available and, when it is, it may not reflect the evolution of an asset's characteristics over time.

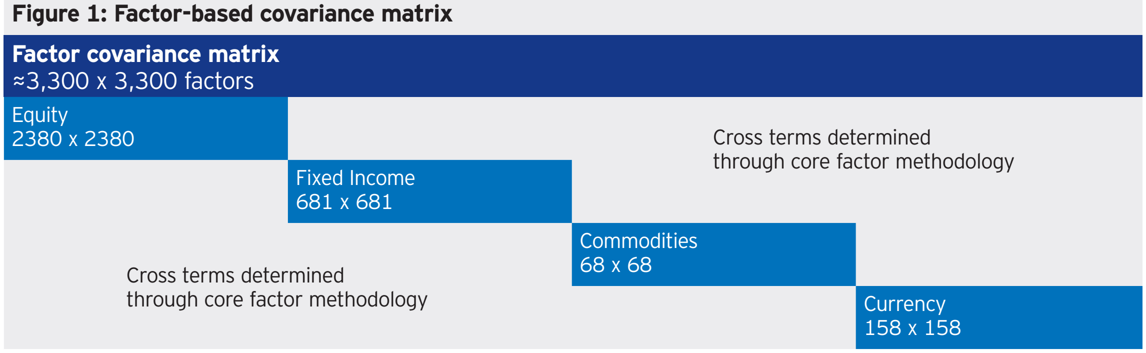
Fundamental factor models do not rely on time series regression for estimating sensitivities but instead use directly observable asset attributes (e.g., industry, price-earnings ratio, price momentum, market capitalization, etc.) to explain returns. These attributes are treated as betas which, when combined with risk indices that correspond to the various attributes identified, allow for the estimation of asset behavior. While all of these risk models have potential benefits, the fundamental risk model provides significant flexibility in practice along with providing for an intuitive understanding of the dependence of an asset's returns on well-defined dependencies. For these reasons, we have selected it to model the risks of the global collection of comprehensive asset classes that are available for use in Invesco Vision. Rather than embarking on the daunting task of developing a proprietary fundamental risk model, Invesco Vision has incorporated BarraOne®, which is one of the most recognized and respected risk models available.

**Modeling traditional assets**

As briefly discussed above, we have chosen a multi-asset fundamental risk factor model to drive our risk modeling capabilities. The model is a comprehensive one utilizing over 3,000 factors, including factors that span all of the major asset classes across various economies, countries, and industries. One of the key benefits of using a factor model is our ability to distill the key performance drivers of any security into a smaller set of relevant systematic factors. This means that, in most cases, we may need to employ only a small subset of relevant factors for modeling exercises.

Figure 1 presents the overall structure of the factor covariance matrix. The matrix is comprised of four subcomponents: equities, fixed income, commodities, and currencies. Equity factors are further subdivided into industry factors and style factors and fixed income factors are further subdivided into term structure and spread factors.

**Figure 1: Factor-based covariance matrix**




Source: Invesco, BarraOne.

Producing the covariance matrix relies on looking at factor return series and determining their variances and covariances. Like most risk models, recent data is weighted more heavily than older data. We offer three variants from which to choose as shown in Figure 2 below. Given the strategic nature of most of our modeling efforts, we will most frequently leverage the longer half-life covariance matrix.

**Figure 2: Covariance matrix time horizon, half-life, and use cases**

| | Variance half-life | Covariance half-life | Use case |
|---|---|---|---|
| Short | 90 days | 2 years | Best suited for tactical positioning |
| Medium | 1 year | 3 years | |
| Long | 8 years | 8 years | Best suited for strategic asset allocation |

Source: Invesco, BarraOne

With a covariance matrix defined, any security or fund can then be modeled through its factor exposures and specific (or idiosyncratic) risk estimates. Figure 3 shows this information for three example securities: a US equity, a EUR denominated bond, and a commodity futures contract. For the US equity example, we can see that the US software firm stock has 100% exposure to the US Software industry factor and additional exposures to a variety of equity style factors which are measured in terms of z-scores. The risk of this stock can then be modeled based on its factor exposures and the underlying factor covariance matrix. Modeling the risk characteristics of portfolios comprised of multiple assets employs the same information.

**Figure 3: Sample equity, bond, and commodity future factor exposures (as of 8/31/2018)**

| Equity: US Software firm | | Bond: European financial firm $4\frac{1}{8}$ 1/14/20 | | Future: Crude Oil Dec 18 | |
|---|---|---|---|---|---|
| **Equity industry** | | **Term Structure** | | **Commodity** | |
| US Software | 100% | EUR Shift | 1.34 | Commodity crude oil shift | 0.96 |
| | | EUR Twist | -2.23 | Commodity crude oil twist | 0.64 |
| | | EUR Butterfly | 1.62 | Commodity crude oil butterfly | 0.17 |
| **Equity style** | | **Spread** | | | |
| US beta | 1.26 | EUR Swap shift | 1.34 | | |
| US Non-linear beta | -1.12 | EUR Swap twist | -2.63 | | |
| US Residual volatility | -0.74 | EUR Swap Butterfly | 0.23 | | |
| US Book-to-price | -0.63 | EUR Financials A | 1.33 | | |
| US Earnings yield | -0.35 | | | | |
| US Dividend yield | -0.14 | | | | |
| US Momentum | 0.92 | | | | |
| US Leverage | -0.26 | | | | |
| US Size | 1.03 | | | | |
| US Non-linear size | -0.53 | | | | |
| US Liquidity | -0.94 | | | | |
| US Growth | -0.20 | | | | |
| **Currency** | | **Currency** | | | |
| USD | 100% | EUR | 100% | | |
| **Specific Risk** | 10.1% | **Specific Risk** | 0.2% | **Specific Risk** | 1.4% |

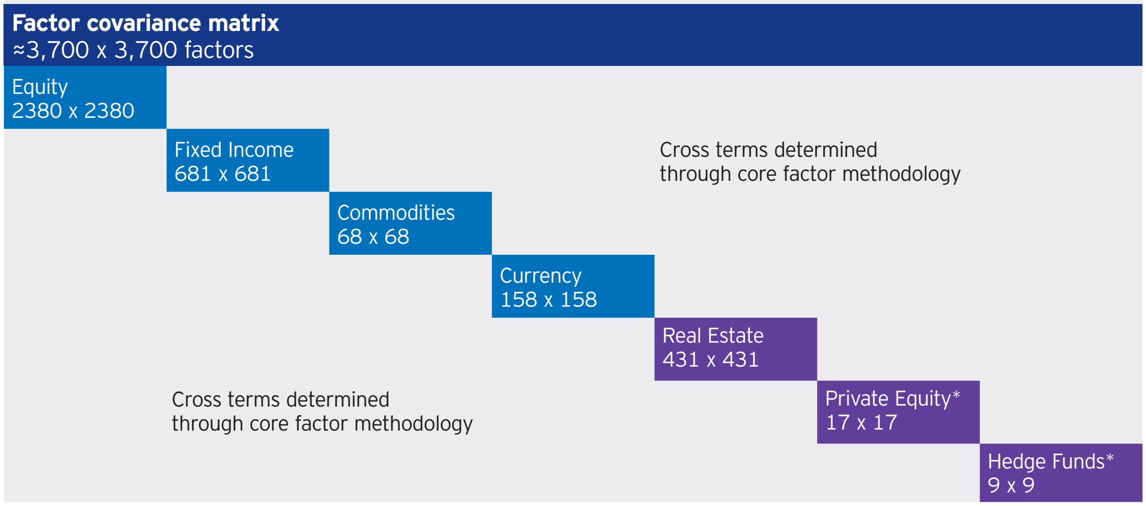
Source: Invesco, BarraOne.

- **Holdings-Based Analysis**
  When the security level holdings of a portfolio are available, we will rely on holdings-based analysis. In this approach, every asset is individually modeled and translated into a set of factor exposures and specific risk estimates, just like the example above. By aggregating the underlying asset exposures, we are then able to effectively model the overall portfolio behavior.
- **Returns-Based Analysis**
  In cases where we do not have information about the underlying portfolio constituents and cannot employ holdings-based analysis, we will proceed with returns-based analysis. In this situation, we rely on the fund's historical returns as well as some high-level fund characteristics, such as asset class, style, and region. We then use a returns-based model to estimate the fund's factor exposures. To do this, a finite set of stylized factor portfolios are employed. The use of stylized portfolios helps address the mismatch between the limited number of historical return data points and the large number of factors in the model. Invesco Vision's model already covers thousands of assets through both holdings- and returns-based analyses. Users can also model a fund on an ad hoc basis using its returns and relevant characteristics.

## Modeling alternative assets

Alternative investments can exhibit unique characteristics that can prove beneficial to achieving desired portfolio outcomes. Given their illiquid nature, modeling alternatives presents additional challenges. Generally, de-smoothing techniques that blend private and public factors to achieve more representative risk characteristics must be applied. Figure 4 presents the structure of the factor covariance matrix with alternative factors included. Detail on specific approaches used for modeling various alternative assets follows.

**Figure 4: Factor-based covariance matrix including alternative factors**




* Private equity and hedge fund assets get exposure to both traditional asset factors as well as the indicated private factors which are uncorrelated to any other factors.
Source: Invesco, BarraOne.

- **Real Estate**
  For private real estate, we model each property based on its property type, subtype, and region. In total, there are 431 real estate factors as shown in Figure 5. Bayesian de-smoothing techniques that include both public and private data are employed to address issues with infrequent, lagged, and non-subjective valuations. When estimating factor exposures, adjustments should consider leverage as indicated by the loan-to-value (LTV) ratio. Furthermore, additional empirical adjustments are made to account for the different behavior of value-add and opportunistic type holdings relative to core holdings. Finally, a specific risk model is also used to capture the idiosyncratic nature of individual holdings.
- **Private Equity and Debt**
  Similar to the real estate model, de-smoothing techniques are employed. However, in this case, assets are assigned exposure to both public and private factors. There is also flexibility to define the public factors to which an asset may be exposed. This allows us, for example, to model a US Technology Buyout fund differently than a European Consumer Cyclical Buyout fund. There are a total of 17 private factors that are delineated by fund type and region as shown in Figure 6.

**Figure 5: Private real estate factors**
Factor matrix shows number of unique regional factors by property type

| Country | Office | Residential | Retail | Industrial | Hotel | Other | Income return | Sub-type | City | Farmland | Agriculture |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Australia | 5 | 1 | 1 | 3 | 1 | 1 | 1 | – | – | – | – |
| Austria | 3 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Belgium | 4 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Canada | 5 | 1 | 5 | 2 | 1 | 1 | 1 | – | – | – | – |
| China | 1 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Czech Republic | 1 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Denmark | 8 | 3 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| France | 6 | 4 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Germany | 8 | 1 | 2 | 1 | 1 | 1 | 1 | – | – | – | – |
| Hong Kong | 1 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Hungary | 1 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Indonesia | 1 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Ireland | 3 | 1 | 5 | 3 | 1 | 1 | 1 | – | – | – | – |
| Italy | 5 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Japan | 7 | 4 | 3 | 1 | 1 | 1 | 1 | – | – | – | – |
| Korea | 5 | 2 | 2 | 2 | 1 | 1 | 1 | – | – | – | – |
| Malaysia | 1 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Netherlands | 4 | 4 | 4 | 4 | 1 | 1 | 1 | – | – | – | – |
| New Zealand | 2 | 2 | 2 | 2 | 1 | 1 | 1 | – | – | – | – |
| Norway | 8 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Poland | 1 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Portugal | 6 | 1 | 1 | 2 | 1 | 1 | 1 | – | – | – | – |
| Singapore | 1 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| South Africa | 3 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Spain | 5 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Sweden | 6 | 4 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Switzerland | 6 | 6 | 2 | 1 | 1 | 1 | 1 | – | – | – | – |
| Taiwan | 1 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| Thailand | 1 | 1 | 1 | 1 | 1 | 1 | 1 | – | – | – | – |
| United Kingdom | 5 | 1 | 3 | 3 | 1 | 1 | 1 | 16 | – | 1 | 1 |
| United States | 4 | 4 | 4 | 4 | 1 | 1 | 1 | 15 | 17 | 11 | 4 |

Source: Invesco, BarraOne.

**Figure 6: Private equity factors**

| Fund type | US | Europe | Asia |
|---|---|---|---|
| Large buyout | ✓ | ✓ | ✓ |
| Small buyout | ✓ | ✓ | ✓ |
| Early stage venture | ✓ | ✓ | ✓ |
| Late stage venture | ✓ | ✓ | ✓ |
| Distressed | ✓ | ✓ | ✓ |
| Mezzanine | ✓ | ✓ | – |

Source: Invesco, BarraOne.

## ■ Hedge Funds

Hedge funds are possibly the hardest of investment types to model. This should be expected as they are very idiosyncratic by nature. Hedge funds are always modeled using returns-based analysis. As a first step, it is necessary to define the hedge fund strategy type. Based on this, a regression is performed against a set of relevant public factors. For a subset of strategies, a hedge fund specific factor is also included in the regression. For example, a merger arbitrage hedge fund will be regressed against the relevant region MSCI IMI index factors, the size factor, as well as a hedge fund "event driven" factor. Figure 7 presents the available hedge fund strategies and corresponding hedge fund and public factors used for modeling exercises.

**Figure 7: Hedge fund factors**

| Hedge Fund Strategy | Hedge Fund Factor | Public Factor (exposure driven by user input, historical fund returns and region) |
|---|---|---|
| Long/Short Equity | – | MSCI IMI; Style factors; Currency factors |
| Dedicated Short Bias Equity | – | MSCI IMI; Style factors; Currency factors |
| Long/Short Credit | – | High Yield; Term structure factors; EM credit spread factors |
| Equity Market Neutral | Equity Market Neutral | MSCI IMI; Style factors |
| Merger Arbitrage | Event Driven | MSCI IMI; Size factor |
| Event Driven Multi-Strategy | Event Driven | MSCI IMI; Size factor |
| Distressed Securities | Event Driven | MSCI IMI; High Yield; Size factor |
| Convertible Arbitrage | Convertible Arbitrage | MSCI IMI; High Yield |
| Fixed Income Arbitrage | Fixed Income Arbitrage | High Yield; Term structure factors; EM Credit Spread factors |
| Global Macro | Global Macro | MSCI IMI; Currency factors |
| Managed Futures | Managed Futures | MSCI IMI; Dollar Index; Currency factors |
| Fund of Hedge Funds | Event Driven, Global Macro, Managed Futures | MSI IMI; Style factors; High yield; EM credit spread factors |

Source: Invesco, BarraOne.

- **Custom alternative assets:**
  While the approaches above can be used to model many alternative assets, there are cases that require more customized modeling. For example, how do we model private infrastructure? How about private commercial real estate debt? While it can be difficult to model these private assets without introducing additional factors, we make the best effort to represent them using the available factors. Given the lack of transparency, limited data, and possible lack of perfect mark-to-market pricing, we choose to keep these modeling exercises as simple as possible.

  For example, for fixed income like alternative assets we may consider rate exposures, credit exposures, and other private factor exposures. Rate exposure is dictated by the nature of the debt. Floating rate debt receives no rate exposure while fixed rate debt includes exposure to the relevant curve shift and twist factors based on the asset's maturity. For credit and private factor exposure, we qualitatively decide which factors are the most relevant. Once we select the factors, we set exposures such that the modeled risk is aligned with our best estimate of the risk of the asset. Finally, the distribution of risk between the public credit and private factors can take on three possible values: 25/75, 50/50, or 75/25. The split is selected through an evaluation of historical correlations with the overall equity and debt markets. In Figure 8 we present examples of various alternative assets and how they might be modeled.

**Figure 8: Sample alternative asset factor exposures**

| Core office building Boston (LTV = 25%) | | Private equity large buyout fund | | Event driven hedge fund | | Infrastructure fixed debt | |
|---|---|---|---|---|---|---|---|
| **Real Estate** | | **Equity Industry** | | **Equity Industry** | | **Term Structure** | |
| US Office east | 133% | US Aerospace & Defense | 3.1% | US Aerospace & Defense | 0.8% | US Shift | 10.0 |
| US Income return | 133% | US Banks | 8.5% | US Banks | 1.3% | US Twist | 6.8 |
| US Boston | 133% | US Biotech life sciences | 4.7% | US Biotech life sciences | 1.0% | | |
| | | US Computer electronics | 5.2% | US Computer electronics | 2.2% | | |
| | | US Diversified financials | 5.9% | US Diversified financials | 2.8% | | |
| | | US Health care equipment and technology | 3.5% | US Health care equipment and technology | 1.5% | | |
| | | ... | | ... | | | |
| | | **Total** | **120%** | **Total** | **30.5%** | | |
| | | | | **Equity Style** | | **Spread** | |
| | | | | US Size | -0.02% | US Swap shift | 10.0 |
| | | | | | | US Utilities BBB | 5.1 |
| | | **Private** | | **Hedge fund** | | **Private** | |
| | | US PE large buyout | 100% | Pure event driven | 73% | US PE mezzanine | 35% |
| **Currency** | | **Currency** | | **Currency** | | **Currency** | |
| USD | 100% | USD | 100% | USD | 100% | USD | 100% |
| **Specific risk** | 8.4% | **Specific risk** | 2.1% | **Specific risk** | 1.3% | **Specific risk** | - |

Source: Invesco, BarraOne.

**Modeling liabilities**

Defeasing a set of liabilities is a common objective for asset managers. Defined benefit plans are managed to support pension liabilities. Insurance companies are managed to meet expected claims. Even retail investors invest with the objective of paying for their future needs. In all these cases, effectively modeling the underlying liabilities is critical.

In order to model a liability stream there are two key ingredients:

1. A cash flow schedule
2. A discount curve

While cash flow schedules are rarely known with certainty, we assume that they are best modeled by the asset owner and as result we treat them as deterministic. For the discount curve we rely on an array of built-in curves based on specific liability characteristics. For nominal cash flows we use nominal yield curves. For inflation adjusted cash flows we use real yield curves. Some curves are based on sovereign rates, others on swap rates, and yet others are based on corporate rates. Figure 9 lists examples of the discount curves used for different types of liabilities:

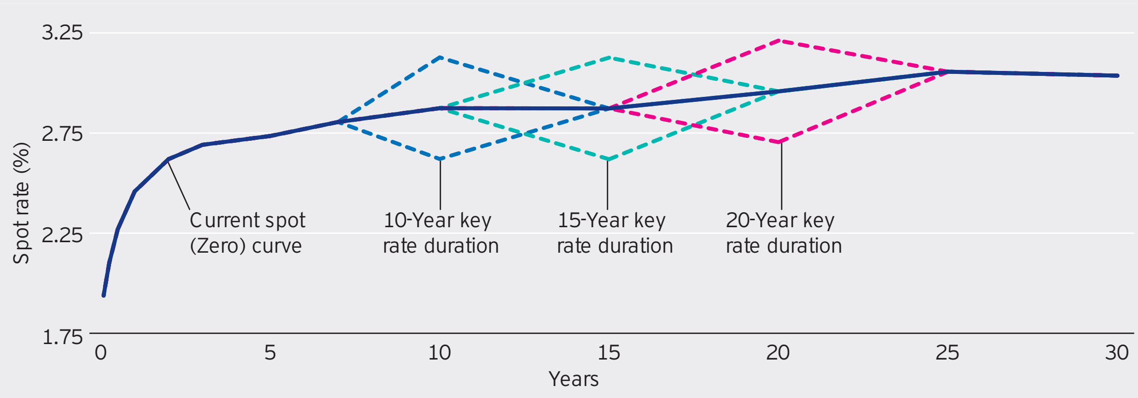
**Figure 9: Liability discount curve variants**

| Liability type | Example curve | Sovereign nominal | Sovereign real | Swap spread | Credit |
|---|---|---|---|---|---|
| US Corporate Pension | Citi AA | ✓ | | ✓ | ✓ |
| UK Inflation-linked liability | Indexed Gilt Curve | | ✓ | | |
| European Insurer | EIOPA EUR | ✓ | | ✓ | |

Source: Invesco, Citigroup, AA Pension discount curve, Bloomberg, EIOPA Solvency II discount curve.

The key aspect of modeling any liability stream is understanding how its present value will react to changes in market conditions. More specifically, we need to translate the liability stream into a set of factor exposures just like those for assets. To do this, we model the liability as if it were a bond with unequal interest payments. Typically, the largest risk entailed in a set of liabilities is interest rate risk – i.e., the impact of interest rate movements on its present value. To better understand this risk, we compute the impact of various interest rate shocks. The shocks we choose to examine are identical to the ones we use when we evaluate the interest rate risk of our bond assets. More specifically, we compute key rate durations at a set of pre-specified key rate points. This is a standard approach that entails re-valuing the liabilities based on "hut like" linear dislocations of the interest rate curve as shown below:

**Figure 10: Key-rate duration calculation methodology**




Source: Invesco, BarraOne.

The exercise requires attention to detail. For example, are we looking for spot key rate durations or par key rate durations? Or does the curve entail a credit component or is it entirely based on a sovereign or swap curve? Seemingly small differences can meaningfully impact the resulting analytics and could lead to unnecessary model risk.

For curves that entail a credit component, such as the FTSE pension discount curve, there is the additional challenge of credits migrating into and out of the curve. To model these types of curves there are several options of varying sophistication that can be employed. Given that we are generally managing multi-asset portfolios, where even small exposures to growth assets may dominate risk, we follow a straightforward approach. Specifically, we model the credit portion of the discount curve using a generic AA corporate spread factor and set the exposure level to be equal to the overall duration of the liability stream.[1]

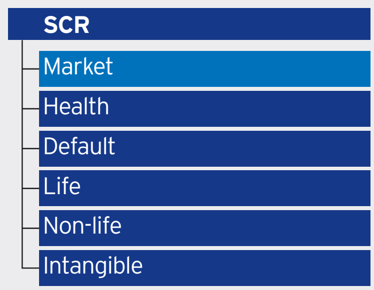
## Regulatory risk models

Invesco Vision also allows insurance entities operating in either the Solvency II framework or the NAIC framework to develop capital-efficient investment portfolios. Depending on the regulatory jurisdiction, insurers must set aside capital as a cushion to protect against adverse movements in their asset portfolios. Each of these frameworks use their own formulaic methodologies for computing the capital charges that will be applied to various asset allocation schemes.

## Solvency II

For Solvency II we focus on the Market Risk component of the Solvency Capital Requirement (SCR) calculation as shown in Figure 11. This is comprised of six sub-components: Interest rate risk, spread risk, equity risk, property risk, concentration risk and currency risk:

**Figure 11: Solvency II Solvency Capital Requirement (SCR) modules**

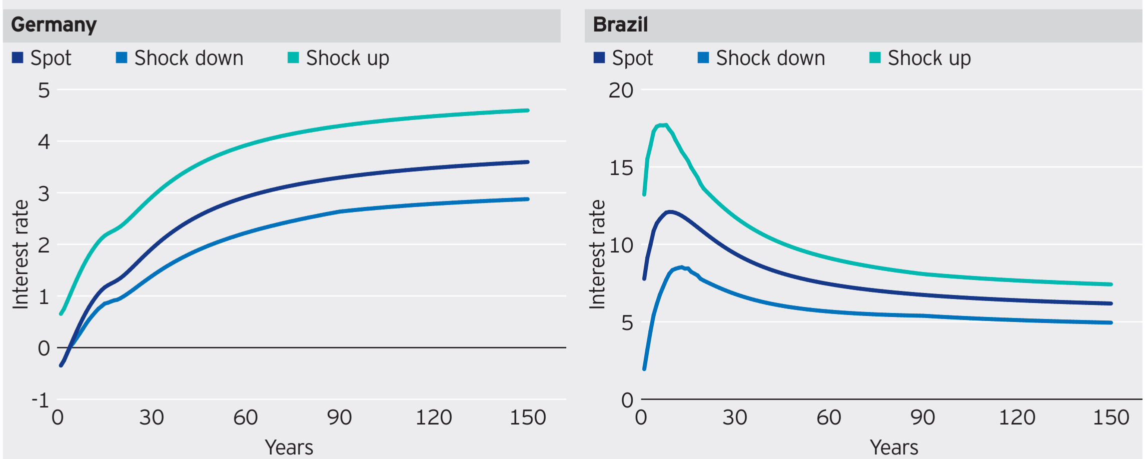



Source: Invesco, Solvency II Directive.

### Interest rate risk:

All interest rate sensitive assets and liabilities are exposed to an upward and downward interest rate shock. The shocks are prescribed by the European Insurance and Occupational Pensions Authority (EIOPA) for various sovereign curves. Two examples are shown in Figure 12.

**Figure 12: Sample EIOPA sovereign rate curves with curve shocks**




Source: Invesco, EIOPA. Data as of Aug. 31, 2018.

1 This approach is used only to estimate the risks of the liabilities. The present value is always computed based on the exact discount curve.

A pricing engine is used to compute the impact of both the upward and downward shock to the assets and the liabilities. This way, each asset and the liabilities have two possible interest rate charges – an upward shock charge and a downward shock charge. Figure 13 shows the resulting charges of various duration liabilities in different regions.

**Figure 13: Example SCR charges for various duration liabilities (%)**

| | USD | | | EUR | | | BRL | | |
|---|---|---|---|---|---|---|---|---|---|
| | Duration | $SCR_{IRup}$ | $SCR_{IRdown}$ | Duration | $SCR_{IRup}$ | $SCR_{IRdown}$ | Duration | $SCR_{IRup}$ | $SCR_{IRdown}$ |
| **SCG 4** | 4.67 | 6.06 | 5.44 | 5.03 | 4.83 | 0.96 | 3.27 | 16.84 | 18.93 |
| **SCG 10** | 12.67 | 12.62 | 11.87 | 13.34 | 12.21 | 5.28 | 7.64 | 27.79 | 37.72 |
| **SCG 16** | 20.67 | 18.77 | 18.44 | 20.85 | 18.43 | 10.83 | 12.93 | 35.30 | 58.25 |

Source: Invesco, EIOPA, BarraOne, Russell standard cash flow generator.

**Spread risk:**
Asset spread SCR charges are computed based on the issuer type, the asset rating, and the asset spread duration. There are three types of assets with spread risk charges: bonds and loans, securitized assets, and derivatives.

- **Bonds and Loans**
  Bonds and loans are further grouped into three categories each with its own treatment: Corporate bonds and loans, infrastructure bonds, and government bonds.
  - **Corporate bonds and loan**
    Corporate bonds and loans entail all bonds that do not fall in the other spread categories. SCR charges for these bonds depend on their rating and spread duration. Figure 14 provides sample SCR charges for specific rating/duration combinations.

**Figure 14: Sample corporate bond and loan SCR charges (%)**

| Spread Duration | AAA | AA | A | BBB | BB | B or less | NR |
|---|---|---|---|---|---|---|---|
| **1-Year** | 0.9 | 1.1 | 1.4 | 2.5 | 4.5 | 7.5 | 3.0 |
| **5-Year** | 4.5 | 5.5 | 7.0 | 12.5 | 22.5 | 37.5 | 15.0 |
| **10-Year** | 7.0 | 8.5 | 10.5 | 20.0 | 35.0 | 58.5 | 23.5 |
| **20-Year** | 12.0 | 13.5 | 15.5 | 30.0 | 46.5 | 63.5 | 35.5 |

Source: Invesco, Solvency II Directive.

  - **Infrastructure Bonds**
    Infrastructure debt is treated more favorably than corporate debt. To qualify for this treatment, the infrastructure project must be located in the European Economic Area (EEA) or Organization for Economic Co-operation and Development (OECD) regions. In the case that the debt is not rated but is senior to all other claims, it will be treated as a BBB rated issue. Any infrastructure projects that are rated below BBB do not qualify. The SCR charge structure is like that of corporates as shown in Figure 15.

**Figure 15: Sample infrastructure bond SCR charges (%)**

| Spread Duration | AAA | AA | A | BBB | BB | B or less | NR |
|---|---|---|---|---|---|---|---|
| **1-Year** | 0.64 | 0.78 | 1.00 | 1.67 | - | - | - |
| **5-Year** | 3.20 | 3.90 | 5.00 | 8.35 | - | - | - |
| **10-Year** | 5.00 | 6.05 | 7.50 | 13.35 | - | - | - |
| **20-Year** | 8.60 | 9.65 | 11.10 | 20.05 | - | - | - |

Source: Invesco, Solvency II Directive.

  - **Government Bonds**
    Bonds issued by the European Central Bank or central governments and banks of member states denominated in local currency are exempt from SCR charges. Bonds issued by other central banks or countries denominated in their local currency do receive SCR charges. Figure 16 presents the SCR charge structure for government bonds.

**Figure 16: Sample government bond SCR charges (%)**

| | European Sovereign Bonds | Non-European Sovereign Bonds | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Spread Duration** | **Any Rating** | **AAA** | **AA** | **A** | **BBB** | **BB** | **B or less** | **NR** |
| **1-Year** | 0.0 | 0.0 | 0.0 | 1.1 | 1.4 | 2.5 | 4.5 | - |
| **5-Year** | 0.0 | 0.0 | 0.0 | 5.5 | 7.0 | 12.5 | 22.5 | - |
| **10-Year** | 0.0 | 0.0 | 0.0 | 8.5 | 10.5 | 20.0 | 35.0 | - |
| **20-Year** | 0.0 | 0.0 | 0.0 | 13.4 | 15.5 | 30.0 | 46.5 | - |

Source: Invesco, Solvency II Directive.

- **Securitized assets**
  Securitized assets are treated punitively in Solvency II. There are three types of securitized assets distinguished generally as Type I, Type II, or Re-securitizations. For an asset to qualify for Type I securitization it must be listed in an EEA or OECD country and needs to also be the most senior tranche. Furthermore, the underlying loans need to be homogeneous and not include re-securitization. Any securitized asset that is not Type I or re-securitized is considered Type II. The SCR charges are based on the type of securitization, the rating, and the spread duration of the underlying asset. Figure 17 presents SCR charges for hypothetical five-year duration securitized assets.

**Figure 17: Sample SCR charges for securitized bonds with a five-year spread duration (%)**

| Spread Duration | AAA | AA | A | BBB | BB | B or less | NR |
|---|---|---|---|---|---|---|---|
| **Type I** | 10.5 | 15 | 15 | 15 | - | - | - |
| **Type II** | 62.5 | 67 | 83 | 98.5 | 100 | 100 | 100 |
| **Re-Sec** | 100 | 100 | 100 | 100 | 100 | 100 | 100 |

Source: Invesco, Solvency II Directive.

- **Derivatives**
  The treatment of credit derivatives depends on whether they are used for hedging or opportunistic bets. Asymmetric absolute and relative shocks should be applied to compute resulting charges.

The total SCR spread charges are computed as the sum of the underlying charges as below:

$$SCR_{SPREAD} = SCR_{bonds} + SCR_{sec} + SCR_{der}$$

**Equity risk[2]:**
There are three types of equity charges as indicated in the following table:

**Figure 18: Equity SCR charges (%)**

| Spread Duration | Description | Charge |
|---|---|---|
| **Type I** | Listed equities in EEA and OECD countries | 39 |
| **Type II** | Listed equities not in EEA or OECD countries and unlisted equities | 49 |
| **Infrastructure** | Qualifying infrastructure equities | 30 |

Source: Invesco, Solvency II Directive.

2 Assets not covered by other modules are also treated as equity Type 2 assets. This includes commodities, alternative investments, and any other assets for which look-through is not possible.

The total equity SCR is then calculated as follows:

$$SCR_{EQUITY} = \left[SCR_{E1}^2 + (SCR_{E2} + SCR_{Einf})^2 + 2 \times 0.75 \times SCR_{E1} \times (SCR_{E2} + SCR_{Einf})\right]^{\frac{1}{2}}$$

**Property risk:**
All direct real estate holdings are charged at a 25% rate. In the case that any leverage is employed it needs to be accounted for by accordingly adjusting the charge.

**Concentration risk:**
Concentration risk is related to having too much exposure in a single entity. This calculation is highly non-linear and must be calculated separately.

**Currency risk:**
All assets that are denominated in a currency that is different from the insurers base currency will be charged at a 25% rate.

Once all the subcomponents of the SCR charges are computed, the total market risk SCR charge can finally be computed as follows:

$$SCR_{MARKET} = \left[\left(\overrightarrow{SCR_{MR}}\right)^* \cdot \Sigma_{SCR}^{\pm} \cdot \overrightarrow{SCR_{MR}}\right]^{\frac{1}{2}}$$

In the above formula, $SCR_{MR}$ refers to the vector of the six SCR charges outlined above and $\Sigma^{\pm}_{SCR}$ refers to the correlation matrix which can take on two possible values as shown in Figure 19.

**Figure 19: SCR Upward and downward correlation matrices**

| | **Upward Shock** | | | | | | **Downward Shock** | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $SCR_{IR}$ | $SCR_{SP}$ | $SCR_{CO}$ | $SCR_{FX}$ | $SCR_{EQ}$ | $SCR_{PR}$ | $SCR_{IR}$ | $SCR_{SP}$ | $SCR_{CO}$ | $SCR_{FX}$ | $SCR_{EQ}$ | $SCR_{PR}$ |
| $SCR_{IR}$ | **1** | 0 | 0 | 0.25 | 0 | 0 | **1** | 0.5 | 0 | 0.25 | 0.5 | 0.5 |
| $SCR_{SP}$ | 0 | **1** | 0 | 0.25 | 0.75 | 0.5 | 0.5 | **1** | 0 | 0.25 | 0.75 | 0.5 |
| $SCR_{CO}$ | 0 | 0 | **1** | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 |
| $SCR_{FX}$ | 0.25 | 0.25 | 0 | **1** | 0.25 | 0.25 | 0.25 | 0.25 | 0 | **1** | 0.25 | 0.25 |
| $SCR_{EQ}$ | 0 | 0.75 | 0 | 0.25 | **1** | 0.75 | 0.5 | 0.75 | 0 | 0.25 | **1** | 0.75 |
| $SCR_{PR}$ | 0 | 0.5 | 0 | 0.25 | 0.75 | **1** | 0.5 | 0.5 | 0 | 0.25 | 0.75 | **1** |

Where:

$SCR_{IR}$ = Interest rate risk charge
$SCR_{SP}$ = Spread risk charge
$SCR_{CO}$ = Concentration risk charge
$SCR_{FX}$ = Currency risk charge
$SCR_{EQ}$ = Equity risk charge
$SCR_{PR}$ = Property risk charge

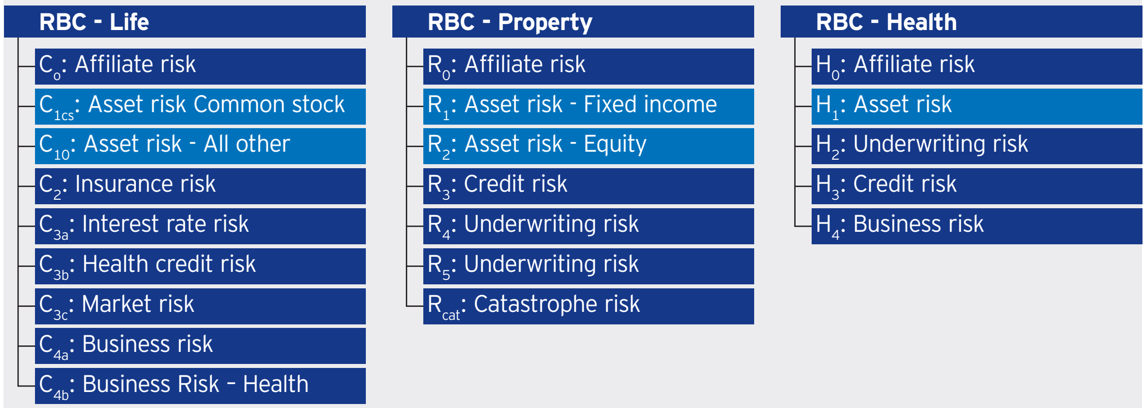
Source: Invesco, Solvency II Directive.

An $SCR_{Market}$ value is computed based on each of the above correlation matrices using the corresponding $SCR_{ir\_up}$ and $SCR_{ir\_down}$ asset and liability charges. The final value of $SCR_{Market}$ is the worst of the two values.

## National Association of Insurance Commissioners (NAIC) Risk-Based Capital (RBC)

Similar to the Solvency II framework, the RBC framework in the US is also comprised of multiple modules. In this case, the modules vary based on the insurance entity type as shown in Figure 20. Invesco Vision addresses only the asset risk modules that focus on investment risks associated with fixed income and equity.

**Figure 20: NAIC RBC insurance company modules**




Source: Invesco, NAIC.

### Fixed income credit risk

This module captures credit related charges. The key driver of the charges is the underlying NAIC designation of the fixed income assets. The charges also vary based on the type of insurance entity as shown in Figure 21.

**Figure 21: Fixed income credit risk RBC charges (%)**

| NAIC Designation | Life (pre-tax) | Life (post-tax) | Property & Casualty | Health |
|---|---|---|---|---|
| **NAIC 1** | 0.40 | 0.30 | 0.30 | 0.30 |
| **NAIC 2** | 1.30 | 0.96 | 1.00 | 1.00 |
| **NAIC 3** | 4.60 | 3.39 | 2.00 | 2.00 |
| **NAIC 4** | 10.00 | 7.38 | 4.50 | 4.50 |
| **NAIC 5** | 23.00 | 16.96 | 10.00 | 10.00 |
| **NAIC 6** | 30.00 | 19.50 | 30.00 | 30.00 |

Source: Invesco, NAIC.

For most fixed income securities, NAIC designations follow a mapping of Nationally Recognized Statistical Ratings Organizations (NRSROs) ratings:

NAIC 1 = A-rated and above
NAIC 2 = BBB-rated
NAIC 3 = BB-rated
NAIC 4 = B-rated
NAIC 5 = CCC-rated
NAIC 6 = Below CCC

### Non-fixed income asset risk

This module includes all non-fixed income assets, such as equity and real estate. Figure 22 outlines these charges for the four types of insurance entities.

**Figure 22: Non-fixed income RBC charges (%)**

| NAIC Classification | Life (pre-tax) | Life (post-tax) | Property & Casualty | Health |
|---|---|---|---|---|
| **Equity** | 30 | 19.50 | 15 | 15 |
| **Real Estate** | 15 | 9.75 | 10 | 10 |

Source: Invesco, NAIC.

Once all the RBC module charges are computed, the total RBC charge can be determined formulaically. Below we indicate the equations for various insurance entities where risks are indicated by the risk modules for each entity type:

$$RBC_{LIFE} = C_0 + C_{4a} + \sqrt{(C_{1o}+C_{3a})^2 + (C_{1cs}+C_{3c})^2 + C_2^2 + C_{3b}^2 + C_{4b}^2}$$

$$RBC_{P\&C} = R_0 + \sqrt{R_1^2 + R_2^2 + R_3^2 + R_4^2 + R_5^2 + R_{cat}^2}$$

$$RBC_{HEALTH} = H_0 + \sqrt{H_1^2 + H_2^2 + H_3^2 + H_4^2}$$

Based on the above equations we introduce the correlation matrices for the two risk categories covered by Invesco Vision in Figure 23.

**Figure 23: RBC correlation matrices for various insurance entities**

| | Life | | Property & Casualty | | Health | |
|---|---|---|---|---|---|---|
| | Fixed income | Non fixed income | Fixed income | Non fixed income | Fixed income | Non fixed income |
| **Fixed income** | **1** | 0 | **1** | 0 | **1** | 1 |
| **Non fixed income** | 0 | **1** | 0 | **1** | 1 | **1** |

Source: Invesco, NAIC.

It is worth noting here a key difference between Solvency II in Europe and RBC in the United States. Whereas Solvency II requires look-through to the underlying holdings of mutual funds, RBC does not. This means an insurer in Europe has the same effective capital charge whether they invest in bonds directly or via a mutual fund; but for US insurers, unrated bond funds will be treated as equity investments, resulting in a much higher capital charge compared to holding the bonds directly.

**Asset level covariance matrix**

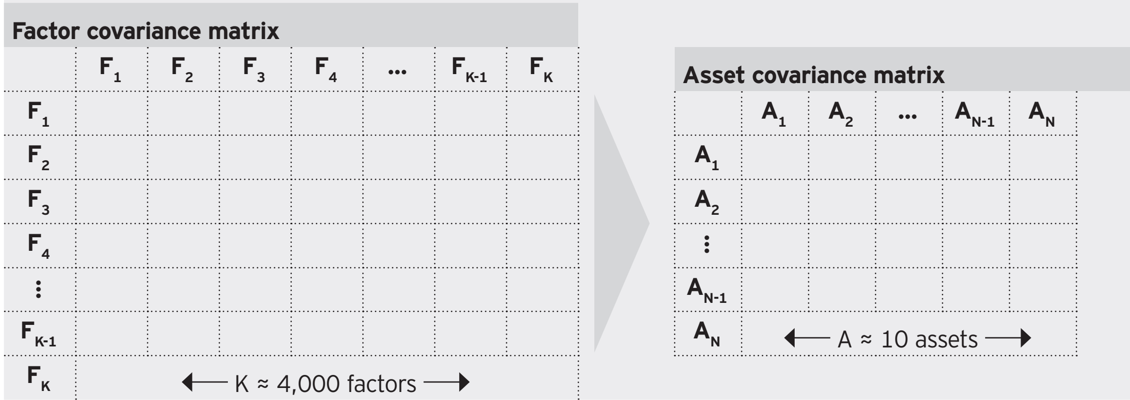
Given either the economic, Solvency II, or NAIC risk factor exposures and their associated risk factor covariance or correlation matrices, we ultimately wish to construct an asset level covariance matrix. We wish to translate the potentially large number of risk factors or capital risk charges into estimates of asset level covariances. For N assets, this results in a compact N by N risk matrix that can be used for portfolio construction and risk estimation exercises.

For the economic risk model, once we have the vector of factor exposures for each asset, we group these vectors into a matrix of factor exposures. We then multiply the factor exposures with the factor covariance matrix to produce the asset level covariance matrix. Namely, given the $K$ by $1$ factor exposure vectors $\beta_j$ *for* $j = 1, 2, \ldots, N$, and the $K$ by $K$ factor covariance matrix $\Sigma_f$, we compute the economic asset level covariance matrix $\Sigma_A$ as follows:

$$\Sigma_A = \left(\beta_1\ \beta_2 \cdots \beta_N\right)^T \Sigma_f \left(\beta_1\ \beta_2 \cdots \beta_N\right) + \Lambda$$

where $\Lambda = diag\left(\sigma_1^2, \sigma_2^2, \cdots, \sigma_N^2\right)$ is a diagonal matrix of idiosyncratic or specific risks. In Figure 24 we illustrate the dimension reduction of the nearly 4,000 by 4,000 factor covariance matrix to an asset covariance matrix.

**Figure 24: Converting factor covariances to asset covariances**




Source: Invesco, BarraOne.

For the Solvency II risk model, we group the solvency risk factors in a similar fashion and compute the matrix product where we now use the 6 by 1 Solvency risk vectors βj for *j =1, 2, …, N* and the 6 by 6 risk up/down correlation matrices $\Sigma^{\pm}_{SCR}$ (corresponding to prescribed up and down interest rate shocks) and define the asset level covariance Solvency II risk covariance matrix as follows:

$$\Sigma_A = (\beta_1 \ \ \beta_2 \cdots \beta_N)^T \Sigma^{\pm}_{SCR} (\beta_1 \ \ \beta_2 \cdots \beta_N)$$

Finally, for the NAIC risk model we construct the asset level risk covariance matrix through a similar process to the one followed for Solvency II. However, for NAIC there are two risk categories and there is only one correlation matrix $\Sigma_{NAIC}$, hence the asset level covariance is formed through the following matrix product:

$$\Sigma_A = (\beta_1 \ \beta_2)^T \Sigma_{NAIC} (\beta_1 \ \beta_2)$$

**Estimating expected returns**

Having established a process to estimate the risk of various assets, we also need to be able to estimate their returns. To do this we rely on Invesco's capital market assumptions (CMAs) that cover a broad number of asset classes across multiple regions of the global economy. However, despite the extensive coverage, there will be cases where our asset blocks do not perfectly align with our CMA asset coverage. As a result, and in order for us to systematically assign returns to any asset block, we have developed a framework that leverages the underlying factor exposures of our CMA and non-CMA assets.

The algorithm aims to create a replicating (minimum tracking error) portfolio of CMA assets for any asset we want to further evaluate. This portfolio is created by leveraging the factor exposures and the relative optimization framework that will be discussed later. We then assume that the replicating portfolio, comprised entirely of CMA assets, should provide a reasonable estimate of the return of the asset in question. The return of the asset is estimated as follows:

$$R_{Asset} = w_1 R_{CMA\text{-}1} + w_2 R_{CMA\text{-}2} + \cdots + w_N R_{CMA\text{-}N} + e$$

where $w_j$ are the CMA asset weights that sum to 100%, $R_{CMAj}$ is the *j*'th forecasted CMA asset's return, and *e* is the residual error.

In most cases, the algorithm is very effective in identifying a CMA asset portfolio that closely tracks the asset being evaluated. However, in some situations, where the asset lies in a space that is not covered by our CMAs, replication can be more difficult. Invesco Vision will alert the user to instances when assets can not be tracked well. In such cases, a thorough, manual review of the estimate is recommended where a return override can be input into the system.

Depending on the specific problem, a user can choose to employ 10-year horizon CMAs or five-year horizon CMAs. Also, for fixed income, it is possible to use the yield as an estimate of return, entirely ignoring the CMAs. Finally, a user may wish to utilize their own CMAs, in which case they would need to input them directly into the system. Figure 25 provides an example of various expected return possibilities.

**Figure 25: Example of expected return selection**

| Asset | Invesco CMA 10-year (%) | Invesco CMA 5-year (%) | Yield / CMA 10-year (%) | Yield / CMA 5-year (%) | User (%) |
|---|---|---|---|---|---|
| US large cap equity | 6.5 | 5.7 | 6.5 | 5.7 | |
| US small cap equity | 7.9 | 7.9 | 7.9 | 7.9 | |
| Europe ex UK equity | 7.3 | 7.4 | 7.3 | 7.4 | |
| UK equity | 8.4 | 8.2 | 8.4 | 8.2 | |
| APAC ex Japan equity | 9.5 | 10.9 | 9.5 | 10.9 | |
| Japan equity | 5.6 | 5.6 | 5.6 | 5.6 | |
| Emerging market equity | 9.5 | 10.4 | 9.5 | 10.4 | |

Source: Invesco.

**Arithmetic versus geometric returns**

In practice, asset returns are most commonly expressed in geometric terms. This is because the investors are most often concerned with either the rate at which an investment grew in the past or the rate it might be expected to grow in the future (or over the long term). The geometric mean return is the average rate of return per period when returns are compounded over multiple periods. Consider a time series of returns $r_t$, for $t = 1, 2, \cdots, T$ periods, and some initial investment amount $W_0$. The value of the investment at time $T$ is $W_T = W_0 \times (1 + r_1) \times (1 + r_2) \cdots \times (1 + r_T)$. The geometric return $\mu_g$, or geometric mean, of such a time series is then:

$$\mu_g = \left( \prod_{t=1}^{T} (1 + r_t) \right)^{1/T} - 1$$

The geometric mean return is of interest to investors because it neatly expresses the periodic growth rate of a time series, i.e., $W_T = W_o(1+\mu_g)^T$. This is of practical importance in terms of understanding the desirability of one investment over another. However, the geometric mean says nothing about risk, or rather, the variability of the returns an investor might actually receive from one period to the next. In fact, two assets can have the same geometric mean but exhibit substantially different variability of returns. To consider risk we must understand the expected value of the return we might receive in any period along with the variability around that expected value. This is where expressing returns in arithmetic terms is useful for investors.

The arithmetic mean $\mu_a$ is just the simple average of the periodic returns produced by an asset over a specified investment horizon and is calculated as:

$$\mu_a = \frac{1}{T} \sum_{t=1}^{T} r_t$$

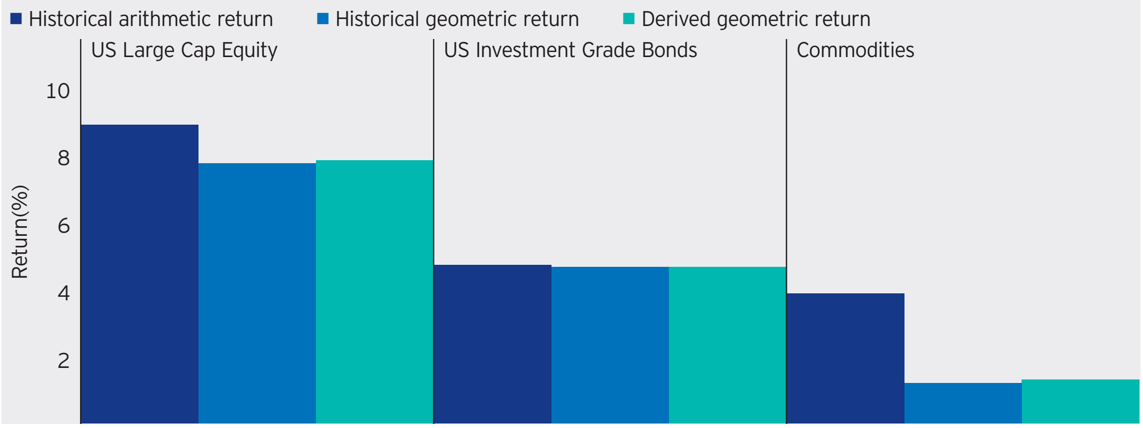
This is particularly important for portfolio construction as it describes the probability-weighted return outcome (central tendency) of a return distribution, or rather, its expected return. If the returns provided by a particular return distribution were all equally likely, then the geometric mean could serve as our expectation. However, returns for most risky financial assets are not equally likely as they exhibit some degree of variability. This variability is most commonly expressed as a function of standard deviation. It can be shown that $\mu_a > \mu_g$ when the standard deviation of a return series is greater than zero. This highlights the fact that the volatility of a return series provides a link between the arithmetic return and the geometric return. Markowitz and Blay (2013) explore various mean-variance approximations to the geometric mean and find that the following approximation provides a reasonable generalization of this relationship:

$$\mu_g = e^{\ln(1+\mu_a) - \frac{\frac{1}{2}\sigma^2}{(1+\mu_a)^2}} - 1 \approx \mu_a - \frac{1}{2}\sigma^2$$

This approximation allows investors to go back and forth between arithmetic and geometric returns as long as they know an asset's or portfolio's arithmetic mean $\mu_a$ and volatility $\sigma$. It should be noted that using the historical information (e.g., arithmetic means, standard deviations, and correlations) in a portfolio analysis will produce portfolios that will have likely performed well in the past. Expected returns should represent expectations for returns that are likely to be achieved in the future expressed in arithmetic terms. The approximation above can also be helpful in producing expected return estimates that are appropriate for use in a portfolio analysis as well as being aligned with intuition in geometric terms.

As an example of how well such a simple approximation can work, in Figure 26 we consider the historical arithmetic and geometric returns for three standard asset classes: 1) US Large Cap Equity, 2) US Investment Grade Bonds, and 3) Commodities and compare the historical geometric return with one derived from the approximation above. The two geometric returns are very close and differ by no more than 10.5 basis points in this example.

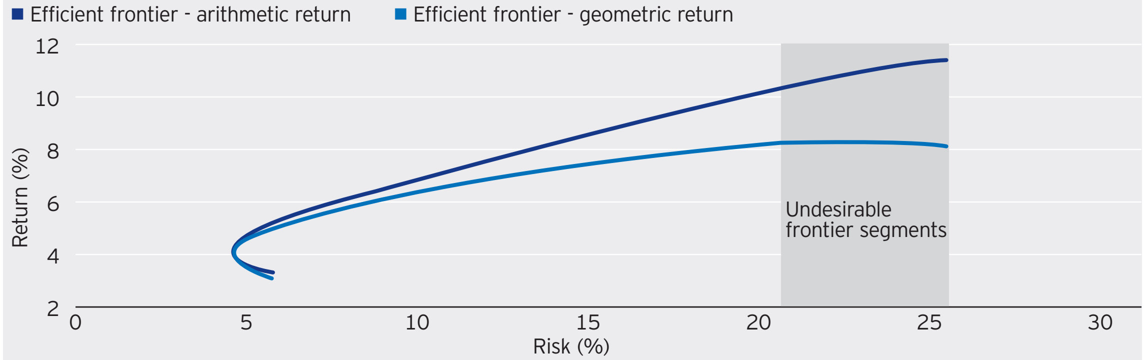
**Figure 26: Historical arithmetic, geometric and derived geometric returns for select asset classes**




Monthly return data period from Sept. 1, 1998 to Aug. 31, 2018.
Note: The historical volatilities of the asset classes over the period are as follows: US Large Cap Equity 14.5%, US Investment Grade Bonds 3.5% and Commodities 22.5%. Past performance cannot guarantee comparable future results.
Source: Invesco, Bloomberg.

The ability to effectively translate arithmetic returns to geometric returns (and vice versa) is of consequence to investors as the return inputs, or expected returns, used in a mean-variance portfolio optimization must necessarily be expressed in arithmetic terms. The reason for this is that the arithmetic mean of a weighted sum (e.g., a portfolio) is the weighted sum of the arithmetic means (of the portfolio constituents). This does not hold for geometric returns. In other words, the weighted average of the arithmetic means of the assets included in a portfolio is equal to the arithmetic mean of the portfolio as a whole. This is not the case when geometric means are used. Since the expected return inputs of a portfolio analysis are required to be in arithmetic terms, the outputs of such an analysis are also in arithmetic terms and must be translated, through the use of the portfolio mean and standard deviation, into the more intuitive geometric terms that describe the expected growth rates provided by the efficient set of portfolios for portfolio selection. Invesco Vision allows for more intuitive portfolio selection by presenting efficient frontiers in geometric terms. Figure 27 presents an example of an efficient frontier presented in both arithmetic and geometric terms.

**Figure 27: Efficient frontier presented in arithmetic and geometric terms**




Source: Invesco.

Note that the efficient frontier expressed in terms of arithmetic returns sits well above the efficient frontier expressed in terms of geometric returns. This is so because the geometric returns are downward adjustments of the arithmetic returns. It is only when we view the efficient frontier expressed in this fashion that we can see how, at segments of the frontier where portfolio volatility is sufficiently large, pursuing portfolios with higher arithmetic returns can result in the likelihood of achieving lower long-term (geometric) returns than portfolios with lower risk. Investors should avoid these segments of the frontier.

**Currency adjusted expected returns**

Portfolios of an international or global nature will likely invest in financial instruments that are based in foreign currencies. For instance, a UK-based multi-asset portfolio manager will likely have an appreciable allocation to US large-cap equities based in USD. Since the UK-based manager wishes to consider their portfolio returns in terms of the local GBP currency there is need to convert the forecasted returns for the US large-cap equity asset class from a USD-based perspective to a GBP-based perspective, especially for the purposes of optimal portfolio construction via mean-variance optimization or its robust counterpart.

For the example UK-based portfolio manager, given an annualized expected return of $\mu_{USD}$ for the USD-based large cap equities, and an annualized US government bond yield of $i_{USD}$ and a similar annualized UK government bond yield of $i_{GBP}$, our formulation for the annualized expected return in GBP is:

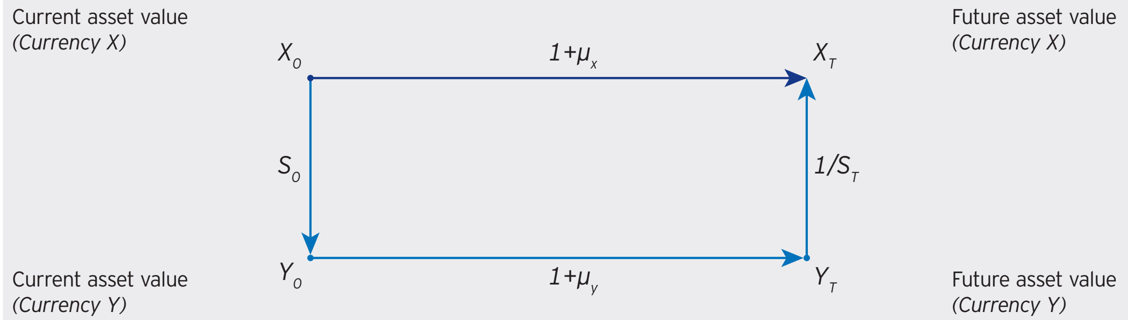
$$\mu_{GBP} = \mu_{USD} - i_{USD} + i_{GBP}$$

In what follows below, we provide the rationale for this return conversion.

At the core of our currency-based expected return conversion process is the concept of Interest Rate Parity. We utilize the basic concept that the future value of an asset denominated in currency *X* is equivalent to the foreign exchange rate-converted future value of the asset denominated in currency *Y*. Figure 28 below graphically depicts such an equivalence.

Specifically, let $X_0$ denote the current value of an asset denominated in currency $X$ and let $X_T$ denote its future value. Then, assuming a single period return of $\mu_x$ the future value is simply $x_T = (1+\mu_x)x_0$. (This is the top dark blue segment in Figure 28.)

**Figure 28: Interest rate parity commutation diagram**




Source: Invesco, BarraOne.

An alternative to going directly from the current value $X_0$ to the future value $X_T$ (in terms its return $\mu_x$ in currency $X$) is to first convert the value of $X_0$ in currency X to the value $Y_0$ in currency $Y$. Such a conversion may be simply expressed as $Y_0 = S_0 X_0$, where $S_0$ is the current foreign exchange rate in going from currency $X$ to currency $Y$. (This is the left-most segment of Figure 28.) Next, assuming a single period return of $\mu_y$, the future value in currency $Y$ is simply $y_T = (1+\mu_y)y_0$ . (This is the bottom segment of Figure 28.) Finally, the future value $Y_T$ may be converted to the future value $X_T$ through a similar foreign exchange rate conversion. Namely, $x_T = y_T/S_T$ where $1/S_T$ is the future foreign exchange rate going from currency $Y$ to currency $X$. (This is the right-most segment of Figure 28.)

Since the future value of the asset denominated in currency $X$ should be the same as the foreign exchange rate-converted future value of the asset denominated in currency $Y$, so as to not violate arbitrage conditions, this means:

$$x_T = x_0(1+\mu_x) = S_0 x_0 (1+\mu_y)(1/S_T)$$

If we perform the same analysis along the same paths, now in terms of two government bonds (whose returns we treat as certain), one denominated in currency $X$ with yield $i_x$ and the other in currency $Y$ with yield $i_y$, then we will have:

$$\frac{1+\mu_x}{1+\mu_y} = S_0/S_T = \frac{1+i_x}{1+i_y}$$

Noting that $(1+\mu_x)(1+\mu_y)^{-1} \approx 1+\mu_x - \mu_y$, and similarly that $(1+i_x)(1+i_y)^{-1} \approx 1+i_x - i_y$, means

$$\mu_y = \mu_x - i_x + i_y$$

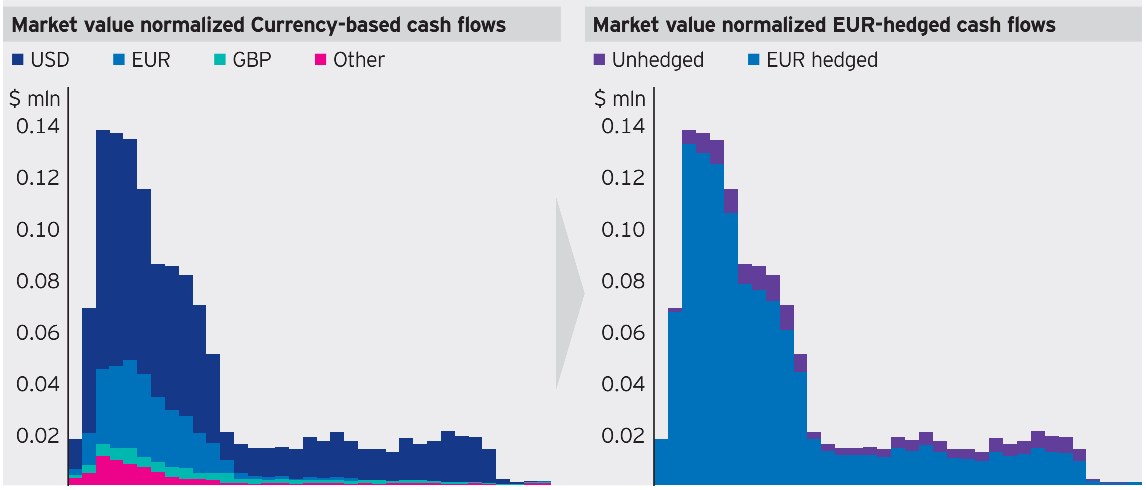
Since our portfolio construction perspective is a strategic, long-horizon one, we use the annualized yields of the 10-year government bonds in currencies $X$ and $Y$ in the above return conversion formula and combine them with the annualized forecasted return in currency $X$. This is our estimate of the forecasted annualized return in currency $Y$. This modeling assumption leads to similar return estimates whether we choose to hedge or not. Of course, from a risk perspective currency hedging will have a meaningful impact.

### Cash flow currency translation

International, or global portfolios, will likely hold investment positions in assets that are denominated in various foreign currencies. For the case of fixed income assets, whose role may be to provide predictable cash flows, this may introduce unwanted risks due to currency and relative interest rate volatility. To address these risks, investors may choose to hedge out the currency exposure using derivatives.

While there are many ways this hedging exercise can be pursued, we assume that it is done using cross currency swaps. Figure 29 provides an example of the result of converting the cash flows provided by the Bloomberg Barclays Global Corporate Index into EUR hedged cash flows.

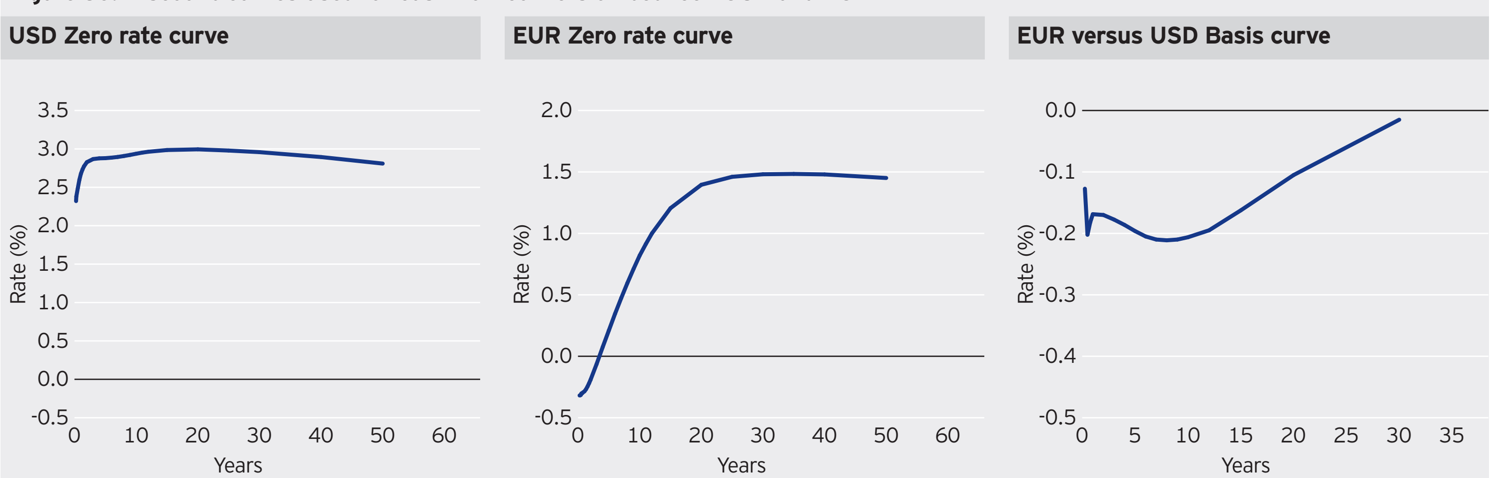
**Figure 29: EUR-hedged equivalent cash flows**




Source: Invesco, Bloomberg.

Conversion between any two currencies requires the use of three distinct curves. Figure 30 shows the three rate curves used for the case of converting a USD-based cash flow into EUR-based cash flows. As a first step we use the USD Zero rate curve to convert fixed USD into floating USD cash flows. We then convert those floating cash flows to fixed EUR cash flows employing the EUR Zero rate curve combined with the basis curve.

**Figure 30: Discount curves used for cash flow conversion between USD and EUR**




Source: Invesco, Bloomberg.

# Portfolio construction

**Absolute risk optimization**
Given a vector of expected annualized arithmetic returns $\mu$ that are determined through our capital market assumption process, and an annualized asset level covariance matrix $\Sigma_A$ that is derived from either our economic, Solvency II, or NAIC risk model, our basic goal in absolute risk optimization is to minimize the absolute portfolio risk for any given portfolio return.

Mathematically, the problem is to find the best collection of investment (or portfolio) weights W* that satisfies:

$$w^* = \underset{w}{argmin} \;\; w^T \Sigma_A w, \text{ such that } \; w^T \mu = \mu^*$$

for a collection of values $\mu_{min} \leq \mu^* \leq \mu_{max}$ . Additionally, we may simultaneously require that each portfolio weight fall within a range of acceptable values, i.e., $a_j \leq w_j \leq b_j$ for $j=1, \cdots, N$ and also require that sum of the portfolio weights meet a required budget, i.e., $\sum_{j=1}^{N} w_j = B$. Typically, we take the value of the budget to be 1 meaning that the portfolio weights add to 100%.

**Relative risk optimization**
Relative risk optimization is a very similar problem to absolute risk optimization, only that in the relative risk case we seek to minimize risk relative to a benchmark or reference asset for any given portfolio return. We treat this problem as being long the portfolio and short the benchmark. Mathematically, the problem may be written as:

$$W^* = \underset{W}{argmin} \;\; W^T \Sigma_{A,\, relative} W, \text{ such that } W^T \delta\mu = \mu^*$$

where the optimal weight vector is now of the form:

$$W = (-1, w_1, w_2, \cdots, w_N)$$

and the return vector is express as being relative to the benchmark:

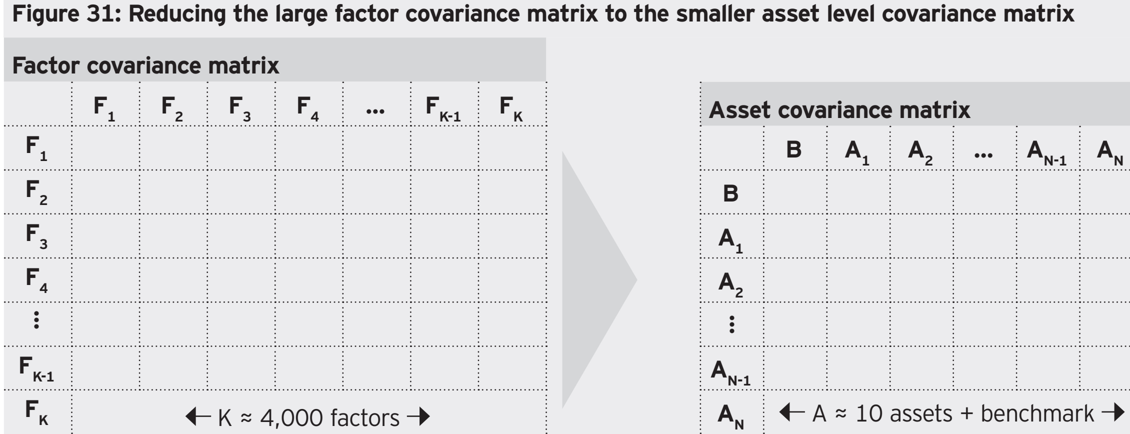
$$\delta\mu = (0, \mu_1 - \mu_b, \mu_2 - \mu_b, \cdots, \mu_N - \mu_b)$$

As in the absolute risk optimization problem, the "$\mu$" is the absolute return for the portfolio and $\mu_b$ is the return of the benchmark of interest. Similar to how we converted the factor covariance matrix into an asset level covariance matrix (see page 17) we create the relative covariance matrix as shown below:

$$\Sigma_{A,relative} = (\beta_b, \beta_1, \beta_2, \cdots, \beta_N)^T \Sigma_f (\beta_b, \beta_1, \beta_2, \cdots, \beta_N) + \Lambda_{rel}$$

where $\Sigma_f$ is the factor covariance matrix, i.e., the matrix of covariances between the risk model's factors and $\Lambda_{rel}$ is the relative specific risk matrix. In Figure 31 we illustrate the translation of the nearly 4,000 by 4,000 factor covariance matrix to the benchmark relative asset covariance matrix.

**Figure 31: Reducing the large factor covariance matrix to the smaller asset level covariance matrix**




Source: Invesco, BarraOne.

## Robust mean-variance portfolio optimization

A mean-variance optimal portfolio is one that minimizes risk for any given portfolio return. However, unconstrained mean-variance optimization (MVO) can exhibit characteristics that are undesirable for some investors. For example, if some of the assets are close substitutes, asset weights can be unstable, i.e., a small change to the expected returns yields substantially different asset weights even though the distribution of returns provided by the portfolio is likely to be only marginally affected.

Additionally, unconstrained MVO can result in portfolio allocations that are highly concentrated in a single asset or across a small number of assets. In this sense, the basic implementation of MVO is not robust to the likelihood of errors in the estimation of expected returns and exposes investors to the possibility of overweighting underperforming assets. For at least these reasons it may be desirable, if not necessary, to modify the basic MVO framework.

Ceria and Stubbs (2006) have carefully considered the fundamental issues addressed above and have reformulated the MVO problem. At the heart of their modified portfolio optimization process, which they call Robust Mean-Variance Optimization, is the further incorporation of the uncertainty of the expected, or forecasted, returns. They start by assuming that the actual returns the portfolio will realize reside within an uncertainty ellipsoid of known size surrounding the expected returns. This is formulated as follows and is also visually depicted in Figure 32.

$$\left(\mu_{realized} - \mu_{expected}\right)^T \tilde{\Sigma}^{-1} \left(\mu_{realized} - \mu_{expected}\right) \leq \kappa^2$$

Where $\kappa^2 = x_n^2(1-\alpha)$ and $x_n^2$ is the inverse cumulative distribution function of the Chi-squared distribution with n degrees of freedom. Finally, $\tilde{\Sigma}$ is the uncertainty covariance matrix, not to be confused with the asset return covariance matrix, $\Sigma_A$.

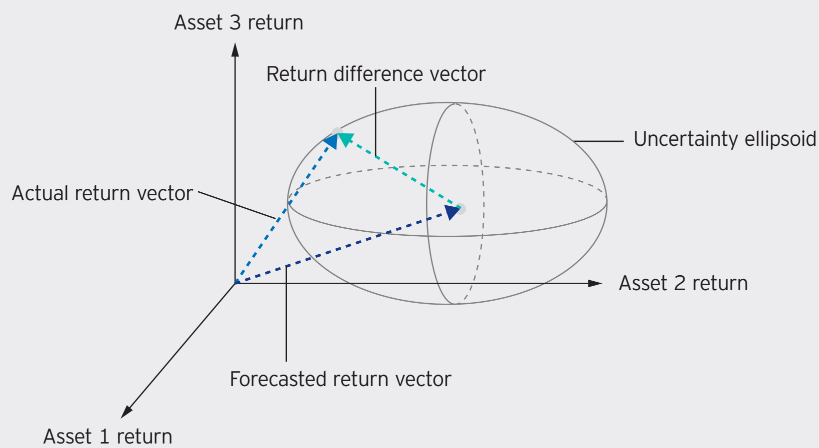
Using the above, the Robust MVO problem can be formulated as follows:

$$w^* = \underset{w}{\operatorname{argmin}} \; w^T \Sigma_A w$$

$$\text{such that } \; w^T \mu_{expected} - \kappa\sqrt{w^T \tilde{\Sigma} w} = \mu^*$$

for a range of candidate values $\mu_{min} \leq \mu^* \leq \mu_{max}$ and require that the asset weights satisfy lower and upper bounds and add to a portfolio weight budget of 100%. In this setting, a penalization term (the second term in the above return constraint involving the square root) has been added to the return target constraint. By incorporating a relatively simple penalty term, the portfolio optimization process can account for expected return uncertainties, becoming less sensitive to small changes in forecasts, and providing optimal asset allocations that are more diversified than those provided by unconstrained MVO.

**Figure 32: Uncertainty ellipsoid showing the distance between the actual and forecasted returns**




Source: Invesco.

### Return agnostic solutions

Given a set of assets, an investor may be interested in identifying possible allocations that are not reliant on return forecasts such as an equal weighted portfolio, an equal volatility portfolio, an equal risk contribution portfolio, a maximum diversification portfolio, or a global minimum variance portfolio.

Each of the above solutions require varying amounts of forecasted information ranging from no information (equal weight), to asset risk estimates (equal volatility), and finally, covariance estimates (equal risk contribution, maximum diversification, and global minimum variance). Figure 33 summarizes the data requirements and the corresponding mathematical formulations for each of the return agnostic solutions.

**Figure 33: Inputs required for various portfolio construction methods**

| Construction method | Volatility forecast | Correlation forecast | Return forecast | Mathematical formula |
|---|---|---|---|---|
| Equal weight | - | - | - | $w_i^* = 1/N$ |
| Equal risk | ✓ | - | - | $w_i^* = 1/\sigma_i$ |
| Equal risk contribution | ✓ | ✓ | - | $w^* = \underset{w}{argmin} \sum_{i,j=1}^{N} \left\| \frac{w_i(\Sigma w)_i}{\sqrt{w^T \Sigma w}} - \frac{w_j(\Sigma w)_j}{\sqrt{w^T \Sigma w}} \right\|$ |
| Maximum diversification | ✓ | ✓ | - | $w^* = \underset{w}{argmin} \frac{w^T \sigma}{w^T \Sigma w}$ |
| Global minimum variance | ✓ | ✓ | - | $w^* = \underset{w}{argmin}\ w^T \Sigma w$ |
| Mean-variance | ✓ | ✓ | ✓ | $w^* = \underset{w}{argmin}\ w^T \Sigma w$ such that $w^T \mu = \mu^*$ |

Source: Invesco, BarraOne.

### Cash flow (liability) matching

An institution or portfolio manager that is expected to make a sequence of future cash payments is faced with a standard liability matching problem. In such a problem there is a well-defined schedule of future cash payments that must be made using the principle and coupon payments from a collection of fixed income investments. For this type of a problem, we seek to create a portfolio, subject to various constraints, that defeases the liabilities at the lowest cost possible.

Cash flow matching problems are rather straightforward in that they rely on linear optimization techniques. Given the universe of available fixed income securities, there is considerable latitude in choosing the subset of the available fixed income securities needed to meet the investment goal. Higher returns are often associated with lower credit ratings, and so a credit rating representing the average (or collective rating) of all of the fixed income securities held in the investment portfolio may be part of the optimization (i.e., minimize cost subject to some desired level of credit quality). Mathematically, we seek to minimize the following objective function:

$$x^* = \underset{x}{argmin}\ p^T x$$

where $x$ represents the vector of amounts of each of the possible fixed income securities to purchase, and $p$ denotes their respective prices. The rest of the problem concerns the formulation of constraints.

Foremost, we must be able to make each anticipated liability payment at the scheduled times $t_1, t_2, \cdots t_K$. Additionally, we may be required to hold a minimum amount of certain fixed income securities such as government bonds and may similarly be required to hold no more than a prescribed amount of higher risk investments such as BBB-rated bonds. This means each investment must be constrained by lower and upper bounds. We also wish to maintain a positive investment balance at each point in time. This means that cash inflows from coupons, repayments, and previous reinvestments should be sufficient to meet cash outflows (liability payments) at any point in time. Lastly, the average or collective credit rating of the portfolio may be required to meet or exceed some pre-specified minimum rating. All of these portfolio attributes can be defined as linear equality or inequality constraints.

First, to meet the liability payments at each time period $t_1, t_2, \cdots t_K$, we impose the following equality constraint

$$(C \quad R)\begin{pmatrix} x \\ b \end{pmatrix} = L$$

where C is a matrix that represents the cash repayments at each point in time for each security. R is a matrix whose entries represent the reinvestment opportunities for positive investment balances through time which takes on the form shown below, b represents the balance of the investment portfolio at each point in time, and L represents the stream of liability payments.

$$R = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 1+r_1 & -1 & 0 & & 0 & 0 \\ 0 & 1+r_2 & -1 & \ddots & 0 & 0 \\ & \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1+r_{T-1} & -1 \end{bmatrix}$$

Second, we require the investment balances all be positive so that $b \geq 0$ for each time period. Third, to meet the requirements of minimum allocations and/or maximum limits to specific instruments, we generically require each asset to satisfy explicit lower and upper bounds. Finally, we impose a minimum portfolio-level credit rating constraint by converting the standard alpha-numeric credit ratings into numeric values.

**Multi-period portfolio construction**

Investors with a long-term investment horizon, during which there may be numerous cash inflows and outflows, face a multi-period portfolio construction problem. In particular, they must consider the totality of the cash flows and their ultimate financial goal in order to determine the best investment strategy through time. Such an optimal investment strategy results in a glidepath or a sequence of time dependent optimal portfolios in which to invest. The strategy can be used to address multiple objectives such as maximizing expected wealth at the end of a 30-year investment horizon subject to various inflows and outflows while not exceeding a specific level of uncertainty.

Modeling such a complex process goes beyond single-period mean-variance optimization and requires one to revisit the foundations of investing in uncertain markets. At its core, the modeling of portfolios in dynamic markets is an exercise in stochastic analysis. For $t=0, 1, 2, \ldots, T$, and cash flows $C = C_1, C_2, \ldots, C_T$, the portfolio's wealth evolves from period t to t+1 as a function of the current portfolio's return $r_{P_t}$ and next period's cash flow through:

$$W_{t+1} = W_t\left(1+r_{P_t}\right) + C_{t+1}$$

Multi-period portfolio construction seeks to maximize the expected terminal wealth $E(W_T|W_0, C)$ subject to pre-specified cash flows while penalizing for the variance of terminal wealth $V(W_T|W_0, C)$. Mathematically, we solve the following optimization problem:

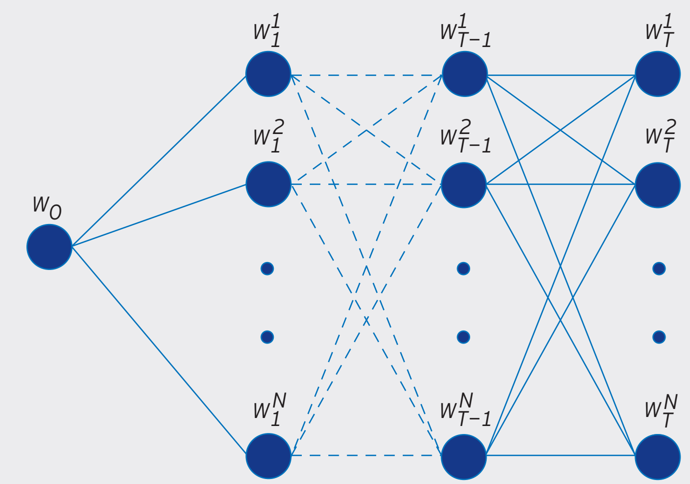
$$\underset{P = P_0, P_1, \ldots, P_{T-1}}{\text{maximize}} \; E(W_T \; W_0, C) - \lambda V(W_T|W_0, C)$$

where $P = P_0, P_1, \ldots, P_{T-1}$ are the portfolios over time. In the equation above, the variance multiplied by a risk aversion parameter $\lambda$. As an example, an investor who is not sensitive to risk will choose $\lambda = 0$ and will effectively maximize expected wealth. On the opposite end of the spectrum, an investor who only cares about risk will choose a very high value of $\lambda$ and will effectively minimize the variance of the terminal wealth. By varying $\lambda$ in the above formulation we are able to create a full efficient frontier.

We conduct the multi-period optimal portfolio construction exercise in the presence of cash flows by following the stochastic dynamic programming (SDP) principle. This means, given a financial goal of interest such as maximizing the probability distribution's mean value and minimizing its variance, we start the optimization process at the end of the investment horizon at time $t = T\text{-}1$ and work backward in time to $t = 0$. Along the way we solve a series of portfolio optimization problems that produce the optimal weights needed to a) achieve the optimal future outcome and b) use these optimal weights to solve the problem in the previous time period.

More specifically, starting at time $t = T\text{-}1$ , we create a grid of N points of future wealth $W_T^1, \ldots, W_T^N$ as well as a grid of N points of current wealth $W_{T-1}^1, \ldots, W_{T-1}^N$. We perform a portfolio optimization process for each of the current nodes in $W_{T-1}^1, \ldots, W_{T-1}^N$ and collect the optimal portfolio weights in the corresponding nodes. Once we have solved the problem at all nodes at time $t = T\text{-}1$, we lock down the optimal weights at the $t = T\text{-}1$ wealth nodes, namely, any portfolio strategy at $t = T\text{-}2$ will use the $t = T\text{-}1$ portfolio weights to reach at $t = T$ portfolio wealth. Once this is determined, we solve the same optimization problem for all the nodes at $t = T\text{-}2$ and continue working backwards until we reach $t = 0$ where we are just solving the optimization problem at only one node, i.e., the starting portfolio wealth $W_0$. Graphically, we represent the SDP strategy by the tree in Figure 34. We note that at each future point in time we can obtain the average of the optimal weights across the nodes to give us the average glidepath.

**Figure 34: The dynamic programming optimization tree**




Source: Invesco.

# Portfolio analytics

**Evaluating factor exposures**

We can decompose the risks within a selected portfolio into various underlying factors. Invesco Vision includes multiple factor groups that allow investors to identify and evaluate the various risks with which they are faced. Figure 36 presents the economic factor groups considered and Figures 37 and 38 present the Solvency II and NAIC factor groups. The following sections provide detail on how to aggregate or decompose factor risks to various levels of granularity or relevant groupings.

**Isolated risk and contribution to risk**

Risk can be decomposed and viewed either in isolation or in terms of contribution to total risk. In both cases, we need to first define the binary group inclusion matrix $M_G$. It is a bit-mask-like diagonal matrix whose entries are either a 1 (meaning inclusion in the group of interest) or 0 (meaning the factor is not included in the group of interest). As an example, consider the "Rates" risk factor group matrix $M_G$ as detailed in the group factor map shown in Figure 35.

**Figure 35: The "Rates" group matrix $M_G$**

| | Factor covariance matrix | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **$Factor_1$** | **$Factor_2$** | **...** | **Shift** | **Twist** | **Butterfly** | **...** | **$Factor_{k-1}$** | **$Factor_k$** |
| **$Factor_1$** | 0 | | | | | | | | |
| **$Factor_2$** | | 0 | | | | | | | |
| ⋮ | | | ⋱ | | | | | | |
| **Shift** | | | | 1 | | | | | |
| **Twist** | | | | | 1 | | | | |
| **Butterfly** | | | | | | 1 | | | |
| ⋮ | | | | | | | ⋱ | | |
| **$Factor_{k-1}$** | | | | | | | | 0 | |
| **$Factor_k$** | | | | | | | | | 0 |

Source: Invesco, BarraOne.

With $M_G$ defined and $\Sigma_f$, the factor covariance matrix, we can compute the isolated risk for a portfolio with $B$ factor exposures and $w$ weights as follows:

$$\sigma_{Group\ Isolated\ Risk} = \sqrt{w^T B^T M_G^T \Sigma_f M_G B w}$$

Similarly, we can calculate the contribution to risk as follows:

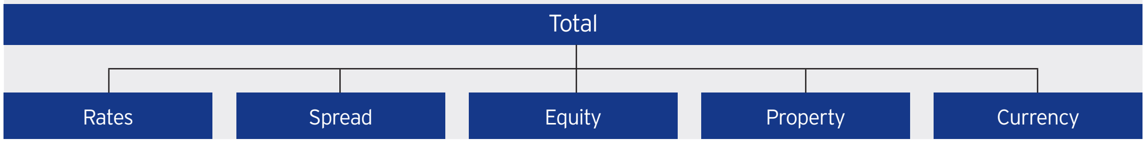
$$\sigma_{Group\ Conrtibution\ to\ Risk} = \frac{w^T B^T M_G^T \Sigma_f B w}{w^T B^T \Sigma_f B w + \sigma_{spec}^2}$$

Here, the group contribution to risk is a percentage of the normalized total portfolio risk, so that all of the group contributions to risk along with the specific risk add up to 100 percent.

Figure 36: Economic factor risk groups

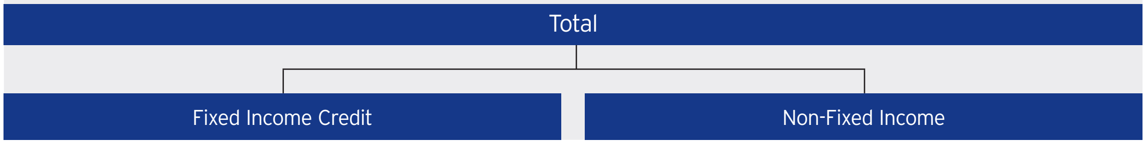

Source: Invesco, BarraOne.

**Figure 37: Solvency II factor group**




Source: Invesco, Solvency II directive.

**Figure 38: RBC factor groups**




Source: Invesco, NAIC.

**Example: Group factor analysis**

We consider a simple example of our factor exposure analysis for a single fixed income instrument, namely an investment grade bond issued by a telecommunications firm. In this case, there are only eight factor exposures as shown below, and the specific risk of the bond is 7.97%.

$$B = \begin{pmatrix} 15.09\ (GOV_{SH}) \\ 13.49\ (GOV_{TW}) \\ 8.53\ (GOV_{BU}) \\ 15.09\ (SWP_{SH}) \\ 11.87\ (SWP_{TW}) \\ 8.80\ (SWP_{BU}) \\ 15.10\ (TEL_{BBB}) \\ 1\ (CUR_{USD}) \end{pmatrix}$$

Assuming that we wish to focus on the interest rate risks (indicated as GOV above), we first proceed to create the bit mask as shown below. Factor entries corresponding to interest rate exposures are indicated with a 1 while the corresponding group factor covariance matrix is shown in Figure 39.

$$M_G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Figure 39: Factor covariance matrix $\Sigma_f$ (values X $10^{-6}$)**

| | $GOV_{SH}$ | $GOV_{TW}$ | $GOV_{BU}$ | $SWP_{SH}$ | $SWP_{TW}$ | $SWP_{BU}$ | $TEL_{BBB}$ | $CUR_{USD}$ |
|---|---|---|---|---|---|---|---|---|
| **$GOV_{SH}$** | 50.63 | 6.82 | -6.61 | -2.00 | 1.45 | 0.06 | -24.18 | 0.00 |
| **$GOV_{TW}$** | 6.82 | 9.22 | 1.67 | -1.16 | -1.65 | -0.75 | 1.27 | 0.00 |
| **$GOV_{BU}$** | -6.61 | 1.67 | 3.35 | -0.27 | -0.56 | -0.76 | 3.82 | 0.00 |
| **$SWP_{SH}$** | -2.00 | -1.16 | -0.27 | 5.57 | 0.79 | 0.23 | -2.55 | 0.00 |
| **$SWP_{TW}$** | 1.45 | -1.65 | -0.56 | 0.79 | 2.91 | 0.42 | -7.78 | 0.00 |
| **$SWP_{BU}$** | 0.06 | -0.75 | -0.76 | 0.23 | 0.42 | 1.25 | -1.66 | 0.00 |
| **$TEL_{BBB}$** | -24.18 | 1.27 | 3.82 | -2.55 | -7.78 | -1.66 | 122.94 | 0.00 |
| **$CUR_{USD}$** | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

Source: Invesco.

Putting the basic ingredients together, we calculate both isolated interest rate risk and interest rate contribution to risk as follows:

$$\sigma_{Group\ Isolated\ Risk} = \sqrt{w^T B^T M_G^T \Sigma_f M_G B w} = 12.21\%.$$

$$\sigma_{Group\ Conrtibution\ to\ Risk} = \frac{w^T B^T M_G^T \Sigma_f Bw}{w^T B^T \Sigma_f Bw + \sigma_{spec}^2} = 25.80\%.$$

**Historical and hypothetical scenario analysis**

An understanding of how a portfolio might have performed historically during various geopolitical and economic environments, as well as how it might perform in certain hypothetical scenarios that could occur in the future, can be used to inform decisions regarding the navigation of potential future market dynamics. Invesco Vision allows for these types of analyses providing detailed decompositions that help to identify key drivers of risk within a portfolio.

- **Historical scenario analysis**

  Modeling a portfolio's performance during a historical period of interest is a straightforward exercise. In general, if one considers a historical period of interest, during which the model risk factors have returns of say $r_{f1}$, $r_{f2}$, through $r_{fk}$, then using the known current exposures each asset has to each of the factors $\beta_{j,k}$, and the weights of each of the assets within the portfolio $W_j$, the portfolio's return for the period can be computed as follows:

$$r_P = \sum_{j=1}^{N} \sum_{k=1}^{K} w_j \beta_{j,k} r_{f_k}$$

It is important to note that this calculation relies on current factor exposures and that these values could have been different during any actual historical period. Historical scenario analysis provides valuable insight and yields the magnitude and direction of the portfolio's return during periods of interest. Invesco Vision covers various pre-determined historical scenarios of interest during which a user can analyze their portfolio and observe its performance, including the 1970s oil crisis, the 1987 market crash, the Global Financial Crisis, and more recently Brexit.

**Example: historical scenario**

We consider a simple historical example for a portfolio holding a single fixed income instrument as we did in the previous section. Here, we examine the 2010 European Bond Crisis spanning March 14 - May 26. During this time, we record the relevant factor returns as well as the corresponding telecommunications firm bond factor exposures as shown in Figure 40.

**Figure 40: The factor exposures, factor shocks and factor returns**

2010 European Bond Crisis using telecommunications firm bond 4.682 '46

| Factors | Exposures | Shocks | Returns (exposure x shock) |
|---|---|---|---|
| $GOV_{SH}$ | 15.09 | 0.35% | 5.27% |
| $GOV_{TW}$ | 13.49 | 0.19% | 2.62% |
| $GOV_{BU}$ | 8.53 | -0.02% | -0.20% |
| $SWP_{SH}$ | 15.09 | 0.03% | 0.51% |
| $SWP_{TW}$ | 11.87 | 0.00% | 0.00% |
| $SWP_{BU}$ | 8.80 | 0.00% | 0.00% |
| $TEL_{BBB}$ | 15.10 | -0.23% | -3.42% |
| $CUR_{USD}$ | 1.00 | 0.00% | 0.00% |
| **Total** | - | - | **4.79%** |

Source: Invesco, Barra One.

Based on the equations shown previously, the portfolio return can be computed as the sum-product of the factor exposures and the factor shocks, which is 4.79%. It is important to note that this calculation is linear in nature and does not include any second order pricing effects such as convexity. When we account for such effects, the return is estimated to be 4.48%.

- **Hypothetical scenario analysis**
  Hypothetical portfolio analysis models various market movements that potentially could happen in the future. There are two types of scenarios: uncorrelated, where changes are isolated to a some specific factor, and correlated, where changes in factors are propagated across all other factors.
  - **Uncorrelated scenario analysis**
    In an uncorrelated scenario analysis with β factor exposures, w weights, and where we assume M factors are shocked by *v*, the portfolio return can be computed as follows:

$$r_P = \sum_{j=1}^{N} \sum_{k=1}^{M} w_j \, \beta_{j,k} \, r_{f_k}$$

  - **Correlated scenario analysis**
    At the heart of correlated scenario analysis is the concept of conditional expectation. We assume that our risk factor returns are distributed as a multivariate normal distribution and that the factor returns are correlated as indicated by the factor covariance matrix. In this case the basic recipe is to 1) prescribe a list of factor shocks, 2) propagate the factor shocks across the remaining factors, and 3) compute the portfolio returns as the sum-product of the factor exposures and the factor shocks.

To propagate the shock, we employ the following calculation:

$$r_f = \Sigma_{2,1} \, \Sigma_{1,1}^{-1} \, r_{f_s}$$

where $r_{fs}$ are the originating factor shocks that are propagated to all factors $r_f$. $\Sigma_{2,1}$ is the covariance between the shocked factors and all other factors and $\Sigma_{1,1}$ is the covariance of the shocked factors as shown in Figure 41.

**Figure 41: The factor covariance matrix block form**

<table>
<tr><th colspan="8">Factor covariance matrix blocks</th></tr>
<tr><th></th><th>$F_1$</th><th>$F_2$</th><th>$F_3$</th><th>$F_4$</th><th>...</th><th>$F_{K-1}$</th><th>$F_K$</th></tr>
<tr><td>$F_1$</td><td colspan="2" rowspan="2">$\Sigma_{1,1}$</td><td colspan="5" rowspan="2">$\Sigma_{1,2}$</td></tr>
<tr><td>$F_2$</td></tr>
<tr><td>$F_3$</td><td colspan="2" rowspan="5">$\Sigma_{2,1}$</td><td colspan="5" rowspan="5">$\Sigma_{2,2}$</td></tr>
<tr><td>$F_4$</td></tr>
<tr><td>⋮</td></tr>
<tr><td>$F_{K-1}$</td></tr>
<tr><td>$F_K$</td></tr>
<tr><td></td><td colspan="2">Shocked factors</td><td colspan="5">← All other factors →</td></tr>
</table>

Source: Invesco, BarraOne.

With all factor shocks having been propagated we can proceed to compute the portfolio returns just as we did in the historical case.

Invesco Vision allows users to consider a collection of hypothetical shocks. The shocks include movements to global equities, US equities, and EAFE equities, US Treasuries, currency exchange rates, oil, and gold. These can be viewed in correlated and uncorrelated terms.

**Example: Hypothetical scenario (uncorrelated and correlated)**
Using the same single asset portfolio in the above examples, we consider the hypothetical scenario in which we shock the US Treasury curve by a parallel 100-basis-point upward movement. As before, there are eight non-zero factor exposures in this example.

In the uncorrelated case, the only factor exposure movements are the ones we have shocked, $GOV_{SH}$, $GOV_{TW}$, and $GOV_{BU}$, and all of the remaining factor shocks are identically zero. In the correlated case, the three explicit interest rate shocks propagate and give rise to non-zero factor shocks across the remaining five risk factors. All of these shocks, along with the factor exposures, are summarized in Figure 42.

**Figure 42: The factor exposures and the uncorrelated and correlated (propagated) factor shocks**

| | | Uncorrelated | | Correlated | |
|---|---|---|---|---|---|
| **Factors** | **Exposures** | **Shocks** | **Returns** | **Shocks** | **Returns** |
| $GOV_{SH}$ | 15.09 | -1.00% | -15.09% | -1.00% | -15.09% |
| $GOV_{TW}$ | 13.49 | 0.00% | 0.00% | 0.00% | 0.00% |
| $GOV_{BU}$ | 8.53 | 0.00% | 0.00% | 0.00% | 0.00% |
| $SWP_{SH}$ | 15.09 | - | - | 0.05% | 0.77% |
| $SWP_{TW}$ | 11.87 | - | - | -0.08% | -0.93% |
| $SWP_{BU}$ | 8.80 | - | - | 0.04% | 0.34% |
| $TEL_{BBB}$ | 15.1 | - | - | 0.63% | 9.44% |
| $CUR_{USD}$ | 1.00 | - | - | 0.00% | 0.00% |
| **Total** | **-** | **-** | **-15.09%** | **-** | **-5.46%** |

Source: Invesco, BarraOne.

Similar to the previous result, the portfolio level return is just the sum-product of the factor exposures and factor shocks. The portfolio return for the uncorrelated and correlated shocks are -15.09% and -5.46%, respectively. When we include full re-pricing that includes the impact of convexity, the returns are -13.58% and -5.26%, respectively.

# Practical application: Case studies

Up to this point we have focused on providing a concise overview of the methods and techniques behind Invesco Vision's capabilities in asset and liability modeling, portfolio construction, and portfolio analytics. We now shift our focus to the practical application of those capabilities. In this section, we present 15 case studies that provide examples of how Invesco Vision can be used to provide insights for better informed investment decision making.

Each of these cases represents a translation of financial theory and quantitative techniques into real-world solutions for real-world investors. We have intentionally provided simple examples that present a particular concept or approach to highlight how Invesco Vision facilitates the application of judgment to identify practical solutions. A key principle used in the development of the system was that relevant information about the risk and trade-offs should be readily available to the user regardless of where they are in the portfolio analysis process. For example, a user can conduct various factor analyses or conduct historical or hypothetical scenario analyses on individual assets, current portfolios, or any point on an efficient frontier while engaged in a portfolio construction exercise…all without being disruptive to the process.

To provide a sense of how the system has been designed we have attempted to reproduce the output from the Invesco Vision interface as closely as possible in all of these cases. You will be presented with information about the cases just as it would be presented in the system. This should provide insight regarding the ease with which various types of information about portfolios can be collected to inform judgment. It should be noted that while these cases focus on presenting the outputs provided by the system, they all rely on inputs that have been shaped by Invesco Vision's asset and liability modeling capabilities. Invesco focuses significant resources toward the continued development and maintenance of these capabilities as they are at the center of portfolio and risk management exercises. Something that may be easily overlooked in reviewing these cases is the extent to which Invesco Vision facilitates the process of turning raw data into useful information that can be used to inform investing decisions.

# Case 1: Absolute risk optimization
## Creating an efficient frontier

Mean-variance optimization is one of the most common approaches used for portfolio construction. The objective of a mean-variance analysis is to produce a frontier of portfolios that are efficient in terms of portfolio mean and variance (or standard deviation). This is an absolute risk optimization exercise where the risk being optimized, is the total risk of the portfolio. Invesco Vision provides this functionality along with capabilities that allow investors to better understand the risk characteristics of each of the portfolio assets, as well as for any portfolio that might be part of the analysis, efficient or otherwise. In this example, we consider an efficient frontier comprised of a small number of fixed income and equity indices. A hypothetical existing portfolio is also included for comparison.

Figure C1a presents the output of an unconstrained mean-variance analysis and is divided into four quadrants. The upper left quadrant presents the efficient frontier, individual portfolio assets, as well as any pre-specified portfolio (commonly an investor's current portfolio) charted along some measure of return and risk. Detailed factor analyses are easily accessible for any point included on this chart. Furthermore, horizontal and vertical projection lines for both the portfolio and benchmark onto the mean-variance efficient frontier provide an easy assessment of possible improvements that could be made in terms of portfolio return and risk.

The upper right part of Figure C1a presents a factor analysis section that provides a deep dive into factor risks associated with a selected investment. If a portfolio or benchmark has been included in the analysis, they are always shown for reference purposes and indicated by the pink (portfolio) and gray (benchmark) dots. The factor analysis section allows users to drill deeper and understand factor exposures and risks in more detail. For example, a user can quantify the isolated risk associated with the equity technology sector or the risk associated with corporate bonds. In addition, users can toggle between three different views: isolated risk, contribution to risk and exposures. Figure C1b provides examples of the latter two views.

The bottom two quadrants of Figure C1a are also dynamic and allow for an array of possible analytics. In this specific example, we have chosen to show portfolio weights that are presented as a frontier composition in the bottom left quadrant. You will notice that the composition chart is aligned with the efficient frontier above. This allows for a quick assessment of how the asset allocation evolves from the lowest risk allocation on the left to the highest risk allocation on the right. Next, the chart in the bottom right quadrant presents the exact asset-weights associated with any point selected on the frontier. In this case, the user's current portfolio is selected. Just like the factor analysis section we again can see the portfolio and benchmark for reference purposes. Users can also conduct historical and hypothetical scenario analyses, review asset correlations, as well as assess cash flows provided by selected portfolios.

While we have presented a simple mean-variance analysis, it serves to highlight the capabilities provided by Invesco Vision that provide support for making better informed investment decisions. Users can evaluate asset and portfolio risks, compare detailed risk and return characteristics of candidate portfolios with current or benchmark portfolios, and run various analyses that provide for a much deeper understanding of the trade-offs presented by investments under consideration. These capabilities facilitate both the portfolio construction exercise as well the ultimate task of selecting a portfolio that is closely aligned with a user's preferences.

**Figure C1a: Absolute risk optimization**
Unconstrained mean-variance efficient frontier

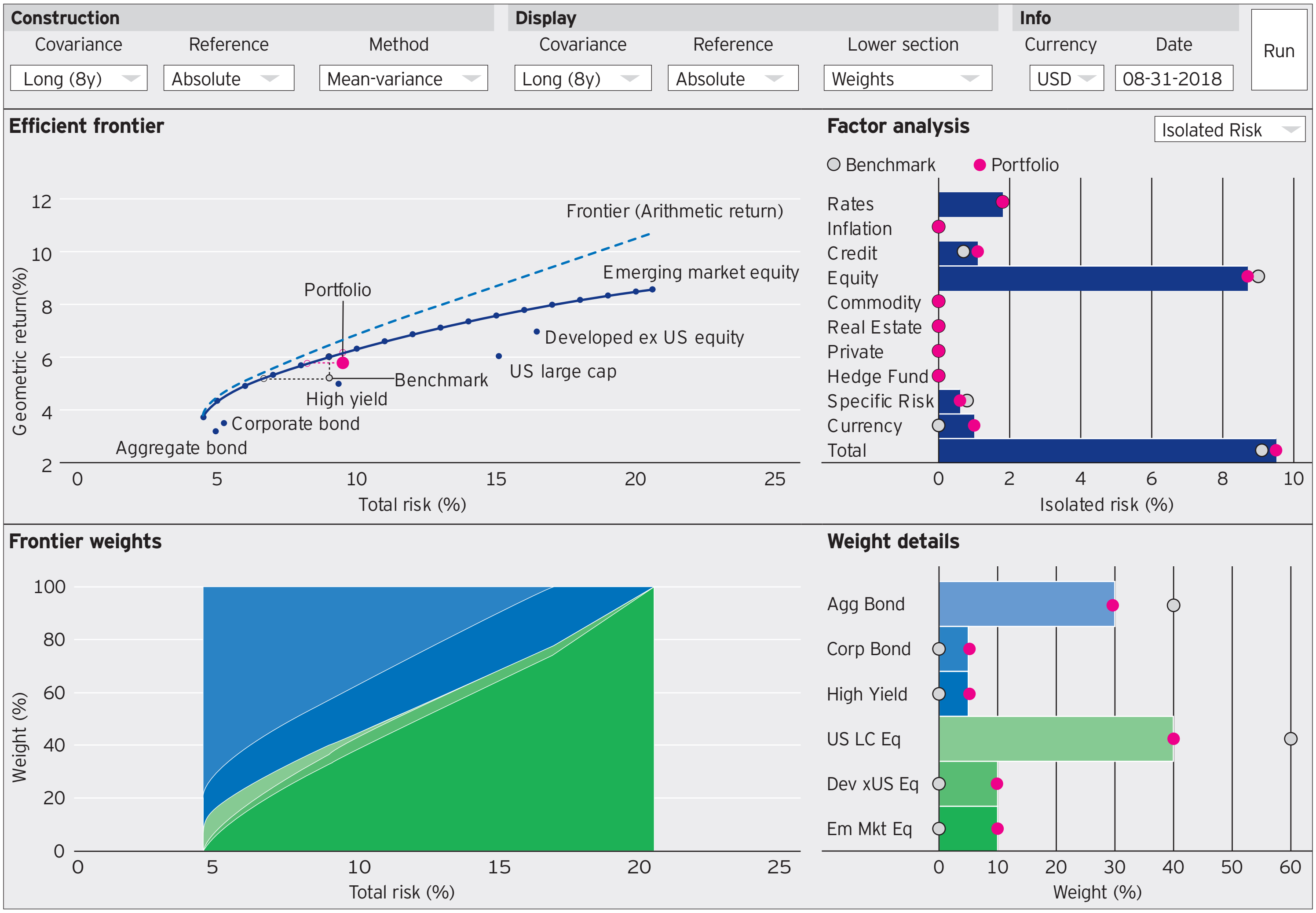
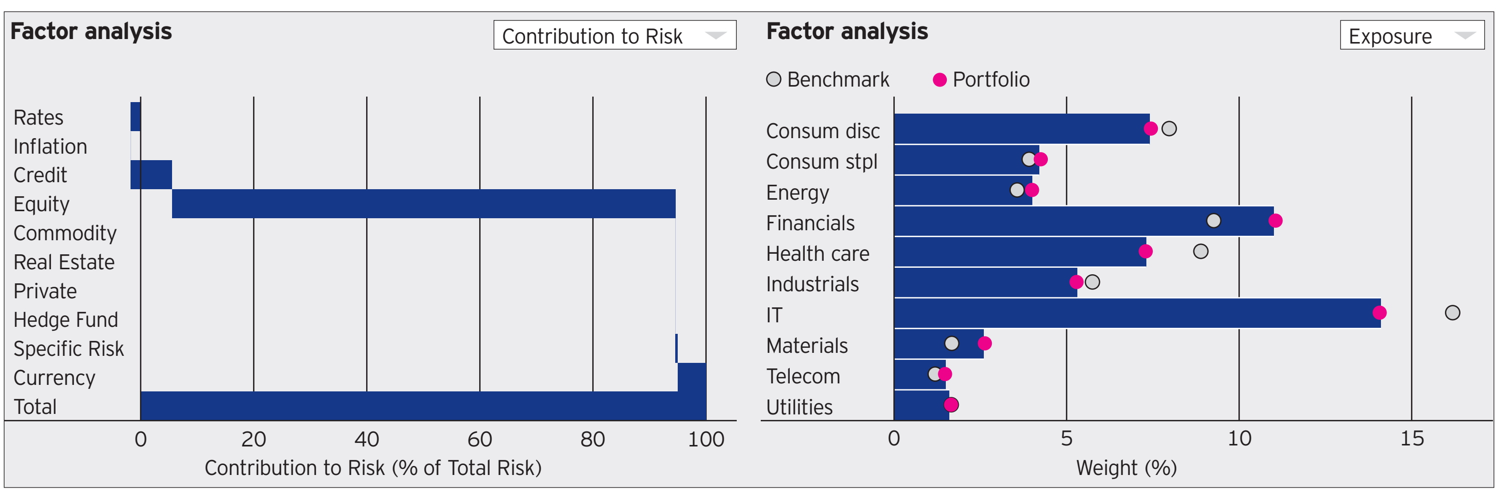



**Figure C1b: Absolute risk optimization**
Portfolio factor analytics



## Case 2: Relative risk optimization

### Creating a style-premia portfolio

While analyzing and understanding portfolios in an absolute risk context is a common starting point for any portfolio construction exercise, it is often the case that portfolios are evaluated relative to some predetermined benchmark. In these cases, we may be interested in creating portfolios where return and risk are considered relative to a reference investment. In this example, we assume we are trying to construct a portfolio with style equity ETFs that will closely track the S&P 500. The rationale for such a portfolio would be to outperform the S&P 500 by efficiently tilting toward various style premia while still tracking the index within acceptable tolerances. In this example, we consider the following equity style ETFs: quality, momentum, high dividend, low volatility, value and size.

In Figure C2b, we provide a first pass at this portfolio construction exercise where we use unconstrained mean-variance to identify our style premia portfolio. You will notice that we have changed the output so that it is presented in relative terms. The ability of decoupling the optimization exercise from how we choose to view the results can at times be important.
For example, we may want to see how a frontier constructed in a relative context looks like from an absolute risk perspective. Here, we see the benchmark (the S&P 500) is placed at zero return and zero risk as would be expected, given that this is a relative optimization exercise. The relative efficient frontier lies above and to the right of the benchmark. This is driven by the higher return estimates for the underlying ETFs. In this instance, the lowest risk (minimum tracking error) portfolio has been selected to present relevant portfolio characteristics. While this portfolio may not have the highest return, it is the portfolio that would be expected to track the S&P better than any other portfolio.

As is indicated in the weight analysis section, this portfolio is comprised of 32.4% quality, 33.0% momentum, 14.1% high dividend, 4.4% low volatility and 16.0% value. Small cap exposure was not included at all. In the factor analysis section, we have drilled into the equity style factors and have switched to the exposure view to better understand our relative style factor exposures. As would be expected, we see several positive loadings that could be the key drivers of any outperformance.

In some cases, we may have priors about the ETFs that have been included or we may want to manually constrain portfolio asset exposures. In Figure C2b we present a second pass at this exercise where we employ the scenario capability to overlay how the frontier would look if we impose the constraint that the portfolio must hold at least 10% of each of the ETFs considered.
As one would expect, the frontier in this example is less efficient. By selecting the lowest risk portfolio on this constrained frontier, we observe in the weights section that the portfolio conforms to the constraints. In addition, the solution has slightly higher relative risk (and lower relative return) compared with the unconstrained solution.

**Figure C2a: Relative risk optimization – Style-premia portfolio**
Unconstrained relative mean-variance efficient frontier

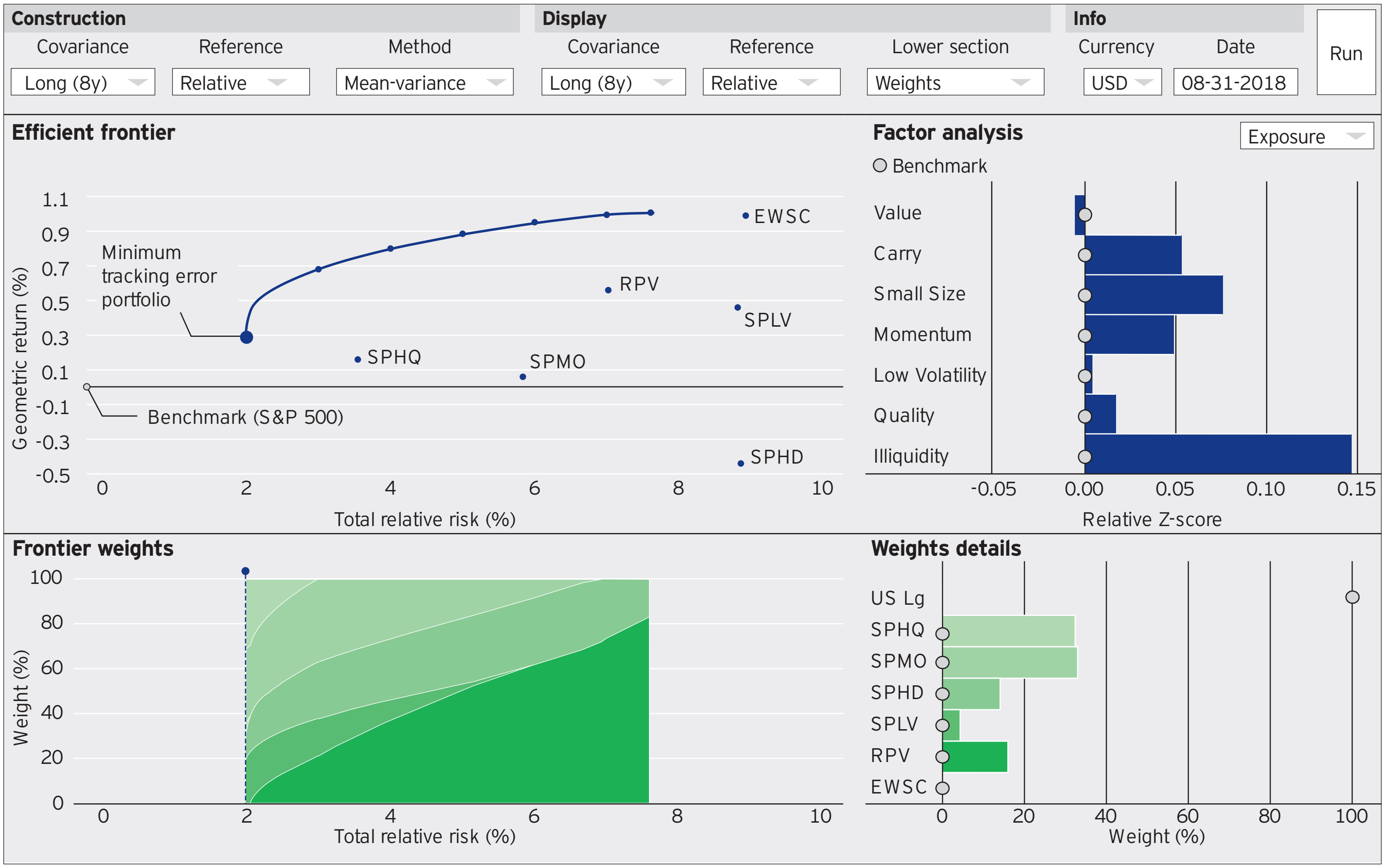
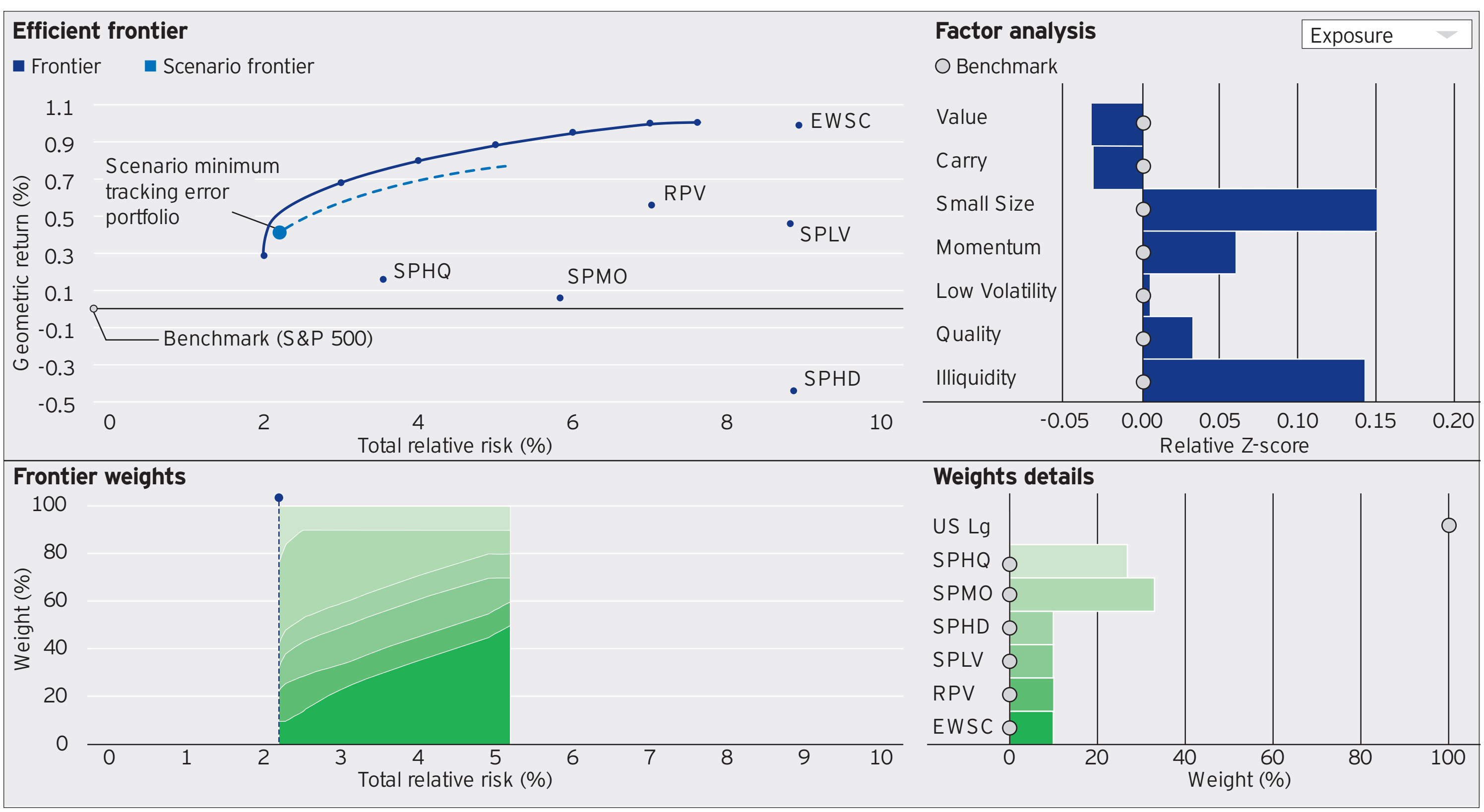



**Figure C2b: Relative risk optimization – Style-premia portfolio**
Constrained relative mean-variance efficient frontier (Requiring at least 10 percent in each constituent ETF)



# Case 3: Relative risk optimization

## Optimizing with a reference portfolio

A reference portfolio is a hypothetical simple diversified portfolio, implemented with passive, low-cost, liquid investments, designed to achieve specific investment goals. Reference portfolios are frequently used by pension plans as a baseline to measure investment performance and to manage risk in the pursuit of returns. The reason for the adoption of a reference portfolio framework is that it provides greater flexibility than bucketed benchmark approaches. In these situations, plans are faced with two options. First, they can construct an allocation and compare it directly to the reference portfolio in an absolute context. Alternatively, plans can develop portfolios using a relative optimization framework. The latter approach offers a few benefits when a plan seeks to avoid performance that deviates substantially from that of the reference portfolio. In this example, we assume the reference portfolio is comprised of 70% MSCI World Index and 30% Bloomberg Barclays Global Aggregate Index. We also assume that the asset owner has a broad array of assets at their disposal as they attempt to deliver higher returns than the reference portfolio. To simplify the exercise, we assume all assets, including the reference portfolio are currency hedged.

Figure C3a shows the reference portfolio and a robust efficient frontier in an absolute risk context. Not surprisingly, based on our return assumptions, the efficient frontier indicates that we are able to produce a portfolio with either higher return for the same amount of risk as the reference portfolio or lower risk for the same amount of return as the reference portfolio. Selecting the higher return option, we notice that the actual portfolio, despite similar absolute risk levels, has meaningfully different factor exposures than the benchmark. This is a direct result of the optimizer finding allocations that have higher return with similar relative risk as it exploits various correlations.

In Figure C3b we consider the problem in relative terms. The reference portfolio now sits at the origin with no risk and no return. The figure also includes two frontiers. The dotted light-blue frontier is the frontier we created in the first figure but displayed on the relative axes. Interestingly, the relative risk (tracking error) of the portfolio we focused on in the top figure is close to 5%. This means that while the portfolio we chose may have similar risk as the reference portfolio, we should still be prepared for meaningful performance deviations. Plan sponsors may not be comfortable with such deviations, despite the higher expected returns.

To address the above challenge, we then go on to create a frontier that is efficient in relative risk terms which is indicated in solid blue. This frontier is more efficient than the absolute frontier with improvements becoming most prominent at lower relative risk levels. More specifically, we notice that we are now able to identify allocations that are expected to track the reference portfolio with less than 1% relative risk. By selecting the lowest risk allocation, we see that it is in fact comprised of an allocation that looks very similar to the underlying constituents of the reference portfolio (i.e., Treasuries, Corporates, MBS, US equities and EAFE equities). While this is encouraging, this point itself may not be of interest, as it also does not offer any excess return to the reference portfolio. However, as we move up the efficient frontier, we are able to identify solutions that are expected to outperform the reference portfolio while minimizing tracking error. For example, if we look directly to the left of the portfolio we evaluated in the context of absolute risk, indicated as the Optimal Absolute Risk Portfolio, we are able to identify a solution with the same return but with 1% lower tracking error.

It is important to note that this type of approach can be applied to other similar types of problems. For example, plan sponsors and corporate entities are often very sensitive to how they are positioned relative to their peers. In such cases, while they may not build their portfolio entirely around what their peers are doing, knowing how they are expected to perform relative to peers can provide meaningful insights and may also lead to a re-evaluation of some of their outlier bets.

**Figure C3a: Absolute risk optimization – Reference portfolio**
Robust optimization without considering reference portfolio

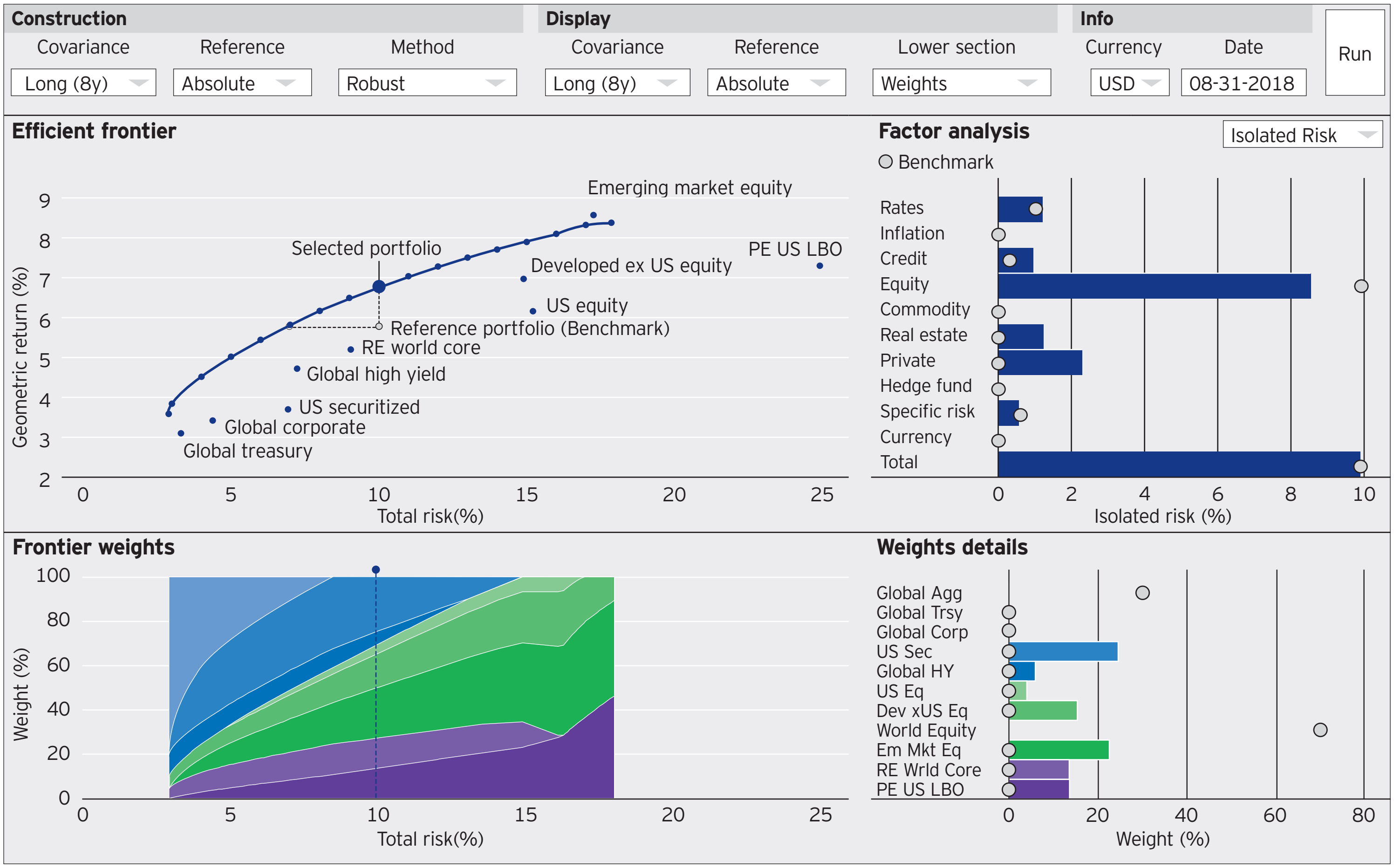
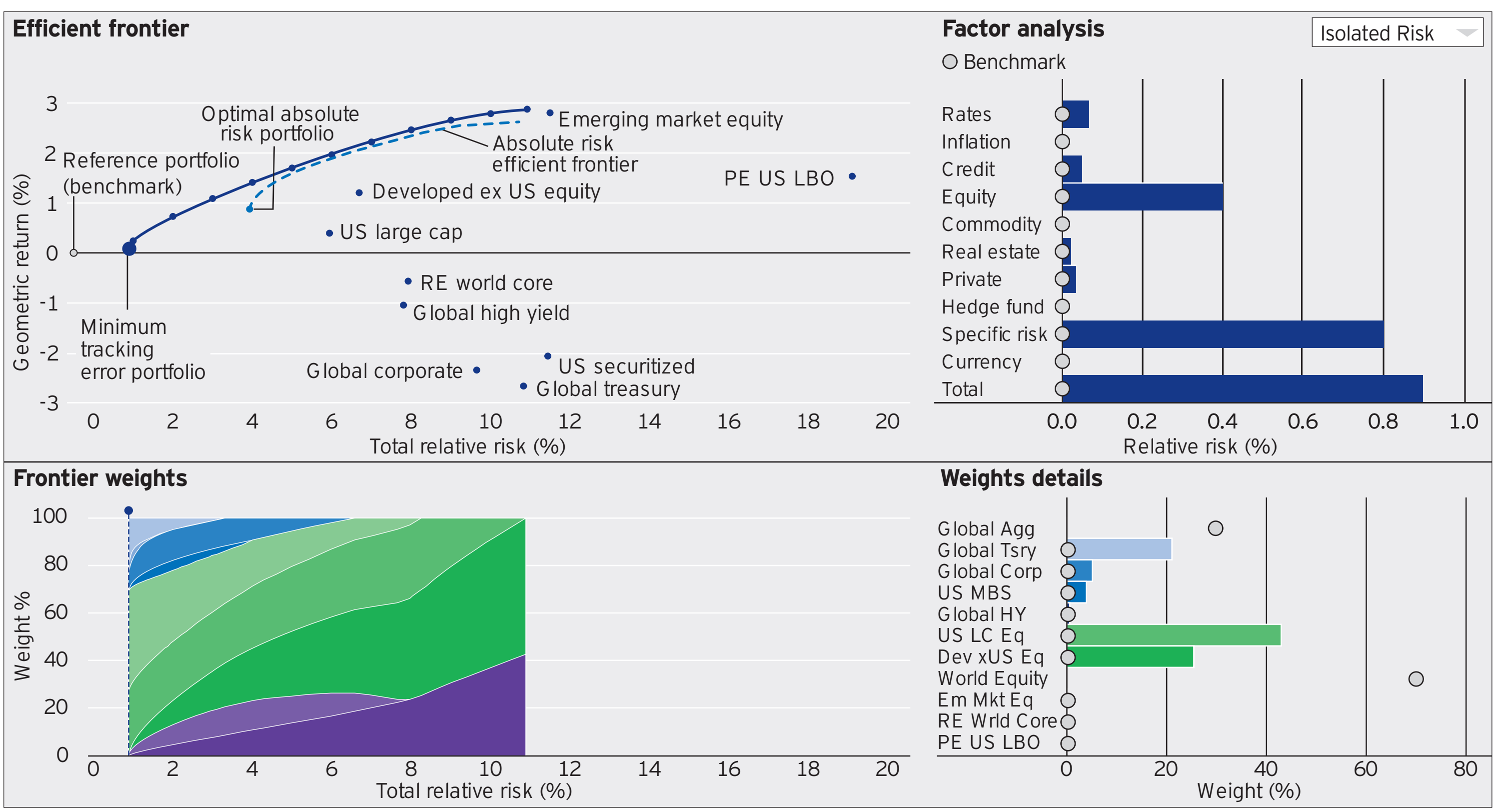



**Figure C3b: Relative risk optimization – Reference portfolio**
Comparison with frontier optimized relative to reference portfolio



# Case 4: Robust optimization

## Addressing estimation error in portfolio construction

A common criticism of unconstrained mean-variance optimization is that it can result in undiversified portfolios or portfolios with large asset concentrations. This behavior is a direct result of mean-variance optimization (MVO) being sensitive to small changes in input parameters. Return forecasts are especially problematic as they are the most influential drivers while also being most likely to be erroneous. To overcome issues with estimation error, practitioners will frequently impose constraints on assets which are believed to be most problematic. While this can produce more diversified portfolios, it can also lead to the application of arbitrary limits on portfolio weights that result in portfolios that aren't necessarily optimal. To achieve more diversified MVO portfolios, we can use the Robust Mean-Variance approach introduced earlier in this paper. In this example, we consider an efficient frontier comprised of a small set of fixed income and equity indices and we compare unconstrained MVO portfolios in Figure C4a with their robust counterparts in Figure C4b. Specifically, we compare efficient portfolios with an expected return equal to that of an included existing portfolio. In both cases, we find that the same return can be achieved at lower levels of risk.

The first, and most notable, point is that the allocations produced through robust mean-variance optimization are far more diversified across assets than those produced by MVO. This is the direct result of robust optimization's explicit incorporation of uncertainty in return expectations. The return uncertainties reduce the dominance of returns in the overall optimization problem. Consequently, while mean-variance optimization allocates virtually all equity exposures to emerging markets equities (EM), robust mean-variance optimization diversifies equity allocations to also include US and developed market equities while still preserving its preference for EM. A similar pattern can be seen in the fixed income investments where corporates are added as part of the allocation at lower risk levels. It also interesting to note how the equal return portfolios produced by each approach differ. Examining the factor analyses for both examples, we see a move from equity factor exposure to more credit and rates. In the robust approach, this is achieved through a more diverse set of underlying assets.

The second point to note is that the efficient frontier is lower and shorter than its mean-variance counterpart. As we seek to diversify away from more concentrated MVO allocations, the resulting portfolios appear sub-optimal given that they include allocations to assets with less desirable characteristics in the MVO sense. However, these portfolios are expected to provide improved out-of-sample performance relative to the theoretically optimal MVO portfolios. Finally, we remark that the length of the robust mean-variance efficient frontier is less than the standard mean-variance efficient frontier. This is also to be expected as the diversification inherent to robust mean-variance optimization limits the ability of the optimal portfolio to invest in any single asset class, as is necessary in this case, to reach the risk and return delivered by the highest risk MVO portfolio which allocates 100% of assets to EM.

A final observation is that the minimum variance portfolios are identical under both optimization frameworks. In both cases, the risk, the return and the underlying asset allocations are identical. This should come as no surprise as the minimum variance portfolio is entirely based on risk and correlations with no dependency on return estimates.

**Figure C4a: Unconstrained mean-variance optimization**
Highlighting frontier portfolio with the same risk as the current portfolio

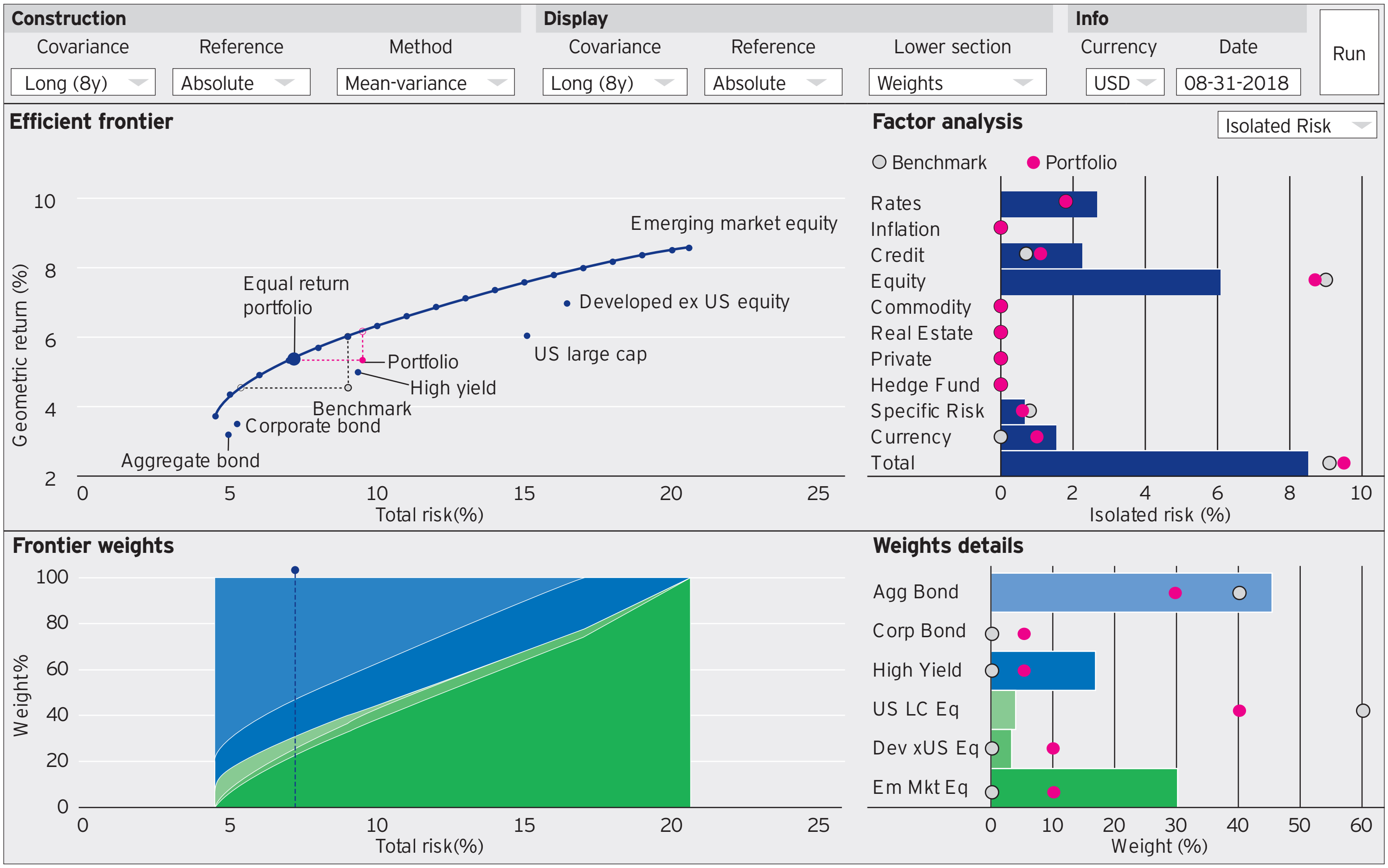
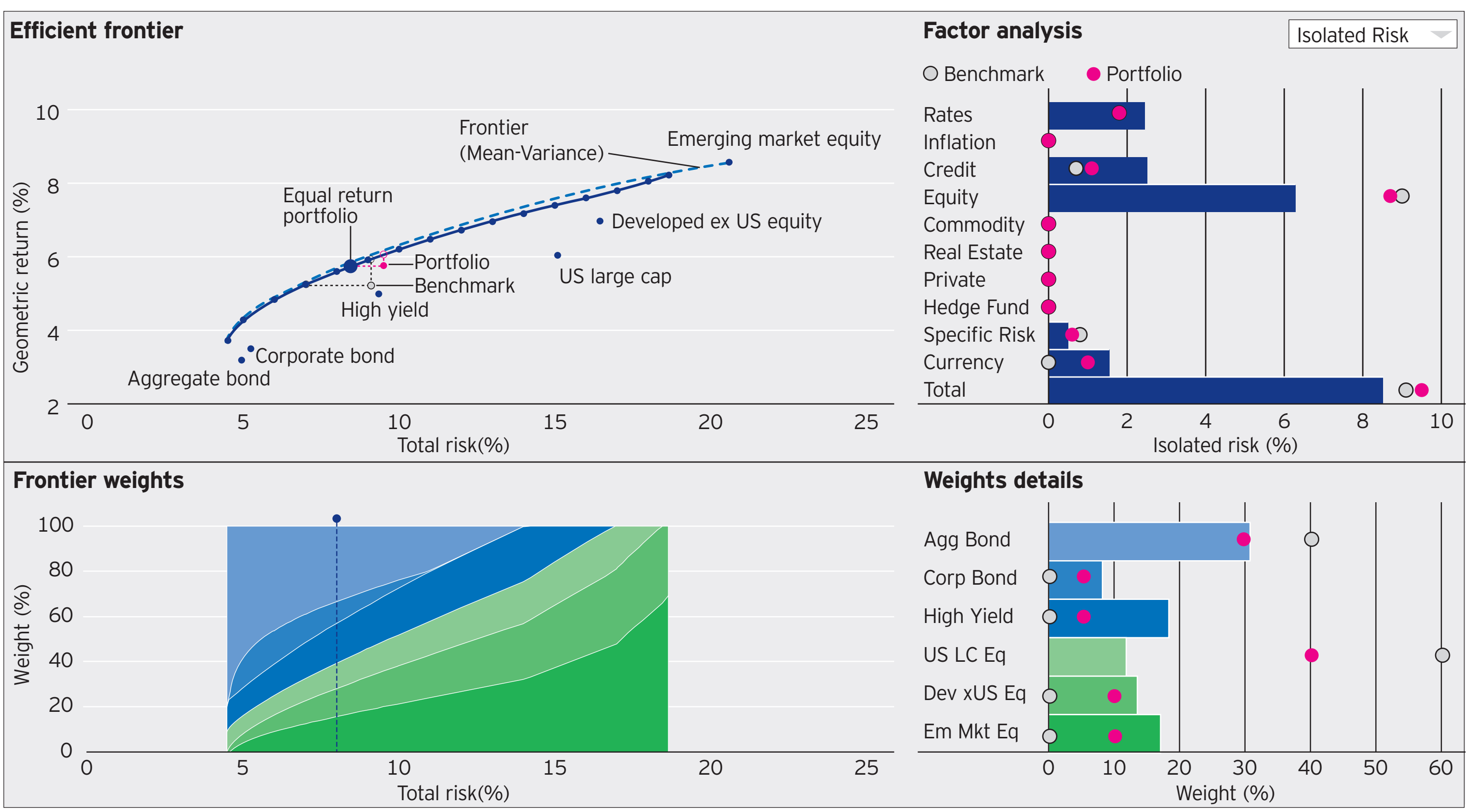



**Figure C4b: Robust optimization**
Highlighting frontier portfolio with the same risk as the current portfolio



# Case 5: Liability-driven investing

## LDI solutions for US corporate defined benefit plans

The first step in any LDI exercise is to obtain a full and detailed understanding of liabilities. In particular, two basic components need to be well understood: 1) the projected cash flows and 2) the discount curve that is used to value them. Once we successfully model the liabilities, we can then use them as a benchmark and employ relative optimization to construct frontiers that will maximize return for any given level of funding ratio volatility. In this example, we will look at a hypothetical US corporate defined benefit plan.

Figure C5a shows the liability worksheet where we can create generic or highly customized liability profiles. In the upper panel we have the choice of creating a generic cash flow schedule using Russell's Standard Cash Flow Generator as well as manually defining a customized set of payments. Once we have defined the cash flow schedule, we can then select a discount curve from a list of various standard options. In this example, we use the FTSE pension discount curve which is commonly used for these types of plans. Finally, one last aspect to consider is liability specific risk. This is most relevant when discount curves have a credit component. Credit migration in these curves leads to risk that cannot be hedged. For a typical liability that is discounted with a AA curve, we will empirically set this value to 2%.

With the necessary components in place, we are then able to model the liability stream. In the right portion of the upper panel, we can see all of the factor risks and exposures. For example, this liability has meaningful interest rate risk, some credit risk and some specific risk. Drill in capabilities allow for a more detailed decomposition of the risks into sub-components such as key rates. We can also toggle between isolated risks, contribution to risk and the factor exposures themselves. Having calculated all the factor exposures, we have effectively translated the liability into a benchmark just like any other third-party benchmark that may now be used as a reference point for further asset allocation modeling exercises. The lower panel of Figure C5a provides additional details that summarize the liability characteristics. For example, the present value of the liability stream is $235 million with only $18 million coming from cash flows beyond 30 years. The yield of the liability is 4.08% which serves as a quick return bogey. The implied spread of the liabilities over Treasuries is 115 bps and the effective duration is 13.7 years, which is close to the highest duration we can get in the physical cash market (excluding Treasury STRIPS).

Having modeled the liabilities, we can go back to the analysis worksheet shown in Figure C5b, where we can begin to focus on asset allocation modeling. For this example, we decide to employ five asset blocks: Intermediate Corporates, Long Corporates, Intermediate Treasuries, Long Treasuries, Treasury STRIPS (15+ years) and the S&P 500. The fixed income blocks have been selected to allow us to introduce varying levels of duration and credit exposure, both of which are key ingredients of the liabilities. The S&P 500 acts as our growth asset. The frontier shown in the left portion of the upper panel indicates portfolios with the highest possible return for a given level of funding ratio volatility. In this instance, we have also chosen to focus on a relatively low risk point on the frontier that is comprised of Long Corporates, Treasuries, and some allocation to the S&P 500. Here, the S&P 500 is what drives the return higher than the liabilities while the fixed income components are reducing the funding ratio volatility by adding duration and credit exposure. In the upper right portion of the panel, we can also see the relative factor risks which are mostly skewed to equity and credit risk as we seek to outperform the liability. The lower panel of Figure C5b overlays the projected cash flows of the assets and the liabilities. As can be seen, there is a cash deficit in the short term and a cash surplus around the 25- to 30-year portion of the schedule. This is the direct result of seeking to minimize duration mismatches with only a fraction of the assets and investing the rest in equities. While appearing somewhat counterintuitive, this is the solution that best minimizes funding ratio volatility while seeking to exploit the benefits of growth assets.

## Figure C5a: Liability-driven investing – US corporate defined benefit plan

Liability analytics

Analyze liability cashflows and discount curves | Load Liability: Search by name or ID | Display: Nominal | Date: 08-31-2018 | Run

### Liability

CFs: 12 | Duration | Curve: USD CitiAA | Select

| Years | Nom CFs | Real CFs |
|---|---|---|
| 0.5 | 9.42 | |
| 1.5 | 10.1 | |
| 2.5 | 10.78 | |
| 3.5 | 11.46 | |
| 4.5 | 12.12 | |
| 5.5 | 12.76 | |
| 6.5 | 13.36 | |
| 7.5 | 13.93 | |
| 8.5 | 14.46 | |

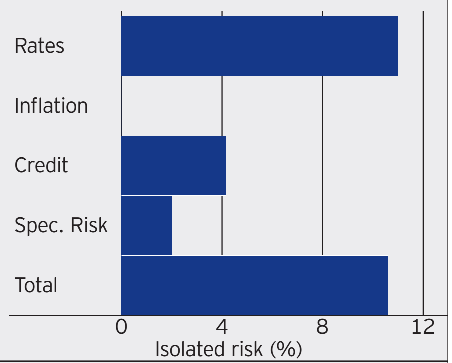
| Years | Nom Rates | Real Rates |
|---|---|---|
| 0.5 | 2.83 | |
| 1.0 | 3.04 | |
| 1.5 | 3.12 | |
| 2.0 | 3.17 | |
| 2.5 | 3.24 | |
| 3.0 | 3.28 | |
| 3.5 | 3.32 | |
| 4.0 | 3.35 | |
| 4.5 | 3.38 | |

Specific Risk: 2

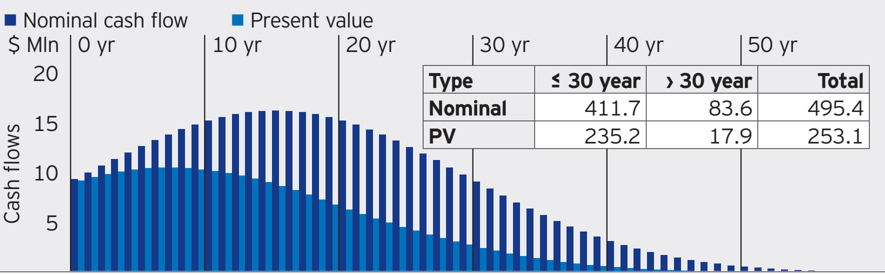
Real CF Multiplier: 1

### Factor analysis

Isolated Risk




### Cash flow




| Type | ≤ 30 year | › 30 year | Total |
|---|---|---|---|
| Nominal | 411.7 | 83.6 | 495.4 |
| PV | 235.2 | 17.9 | 253.1 |

### Analytics

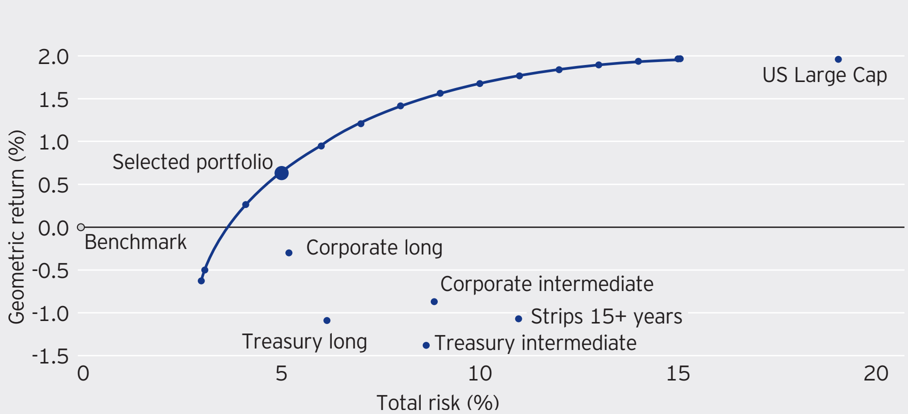
| Items | Nominal |
|---|---|
| Yield (%) | 4.08 |
| OAS to treasury (bps) | 115 |
| Effective duration | 13.72 |
| Modified duration | 14.13 |
| Convexity | 2.98 |

## Figure C5b: Liability-driven investing – US corporate defined benefit plan

Liability relative efficient frontier

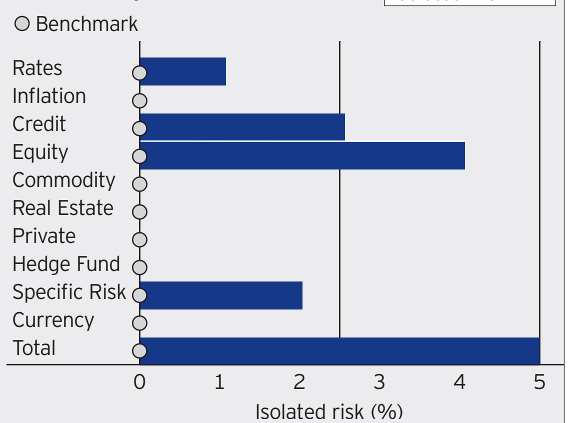
| Construction | | | Display | | | Info | | |
|---|---|---|---|---|---|---|---|---|
| Covariance | Reference | Method | Covariance | Reference | Lower section | Currency | Date | Run |
| Long (8y) | Relative | Mean-variance | Long (8y) | Relative | Cash flows | USD | 08-31-2018 | |

### Efficient frontier

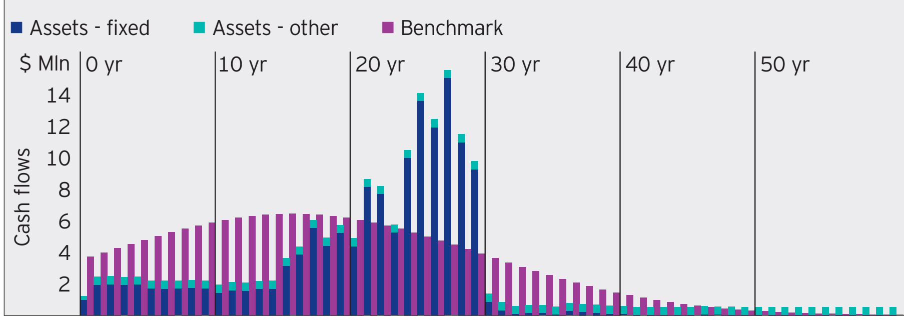



### Factor analysis

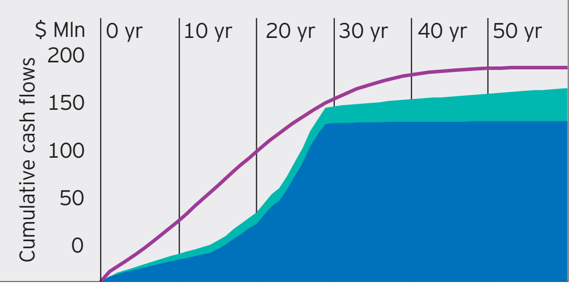
Isolated Risk




### Frontier weights




### Weight details



# Case 6: Liability driven investing

## LDI solutions for UK defined benefit plans

Working with UK defined benefit plans has many similarities, as well as some differences, compared to US and other global oriented liability exercises. On the similarity side, we still need two key ingredients: 1) the projected cash flows and 2) the discount curves used to value them. Regarding the more unique characteristics, UK DB plans will also include inflation-linked cash flows in addition to the typical nominal cash flows as dictated by the UK pension system. To address this, we allow for both types of cash flows and also introduce the corresponding required real rate curves.

Figure C6a shows the liability worksheet where we can enter both nominal and inflation linked cash flow streams. We then need to select the corresponding discount curves. In most situations this will be the nominal and real Gilt curves. In addition, and mostly for research purposes, we introduce a "multiplier" option that can be used to scale the relative magnitude of the two types of cash flows. For UK plans, the specific risk component is not relevant since the discount curves are based on sovereign yield curves that are not susceptible to the credit migration risks associated with the US corporate discount curves.

The right portion of the upper panel of Figure C6a summarizes all of the resulting factor risks. As can be seen, the liabilities have both nominal and real rate exposure with no credit exposure or specific risk associated with them. Drill in capabilities allow us to further analyze the liability exposures in terms of nominal and real key rate durations. In this example, the liabilities have more real rate risk than nominal rate risk. This is primarily driven by the longer duration of the real cash flows. The lower panel depicts the underlying cash flows including their future values and present values. Future values are further decomposed into those originating from the nominal cash flow and the real cash flows. Since the real cash flows are expected to grow with inflation, a "real cash flow expected inflation" component needs to also be added and is computed based on break-even rates. Finally, the analytics section of the lower panel summarizes the key liability characteristics. The analytics are presented separately for each type of cash flow. For example, the yield of the nominal cash flows is 1.65% while the real yield of the inflation-linked cash flows is -1.59%. Similarly, the duration of the nominal cash flows is 13.6 years while the duration of the inflation-linked cash flows is 17.2 years.

With the liabilities fully modeled, in Figure C6b we look at how we can create efficient portfolios relative to the liabilities themselves. In this example, we create a simple asset universe comprised of the Intermediate Gilt Index, the Long Gilt Index, the Inflation linked Gilt index and MSCI UK equities. Observing the underlying asset allocation along the frontier we notice that the lowest risk solution is comprised of a blend of Long Gilts, Intermediate Gilts and inflation linked Gilts. This should come as no surprise as the optimizer seeks to most closely match the liability exposures with the provided blocks. It is of interest to note is that the lowest risk solution has lower tracking error than what we saw in the US plan case. This is driven by the discount curve not including a spread exposure, making it easier to more closely match liabilities. Finally, as we move along the efficient frontier we gradually see the Intermediate Gilt block is phased out. This is a direct result of the optimizer opting for more duration in the Long Gilt block as the overall allocation to fixed income is moved into equity. It is important to note that in a more real-world situation, derivatives such as interest and inflation swaps could be used to further improve the depicted efficient frontier.

**Figure C6a: Liability-driven investing – UK defined benefit plan**
Liability analytics

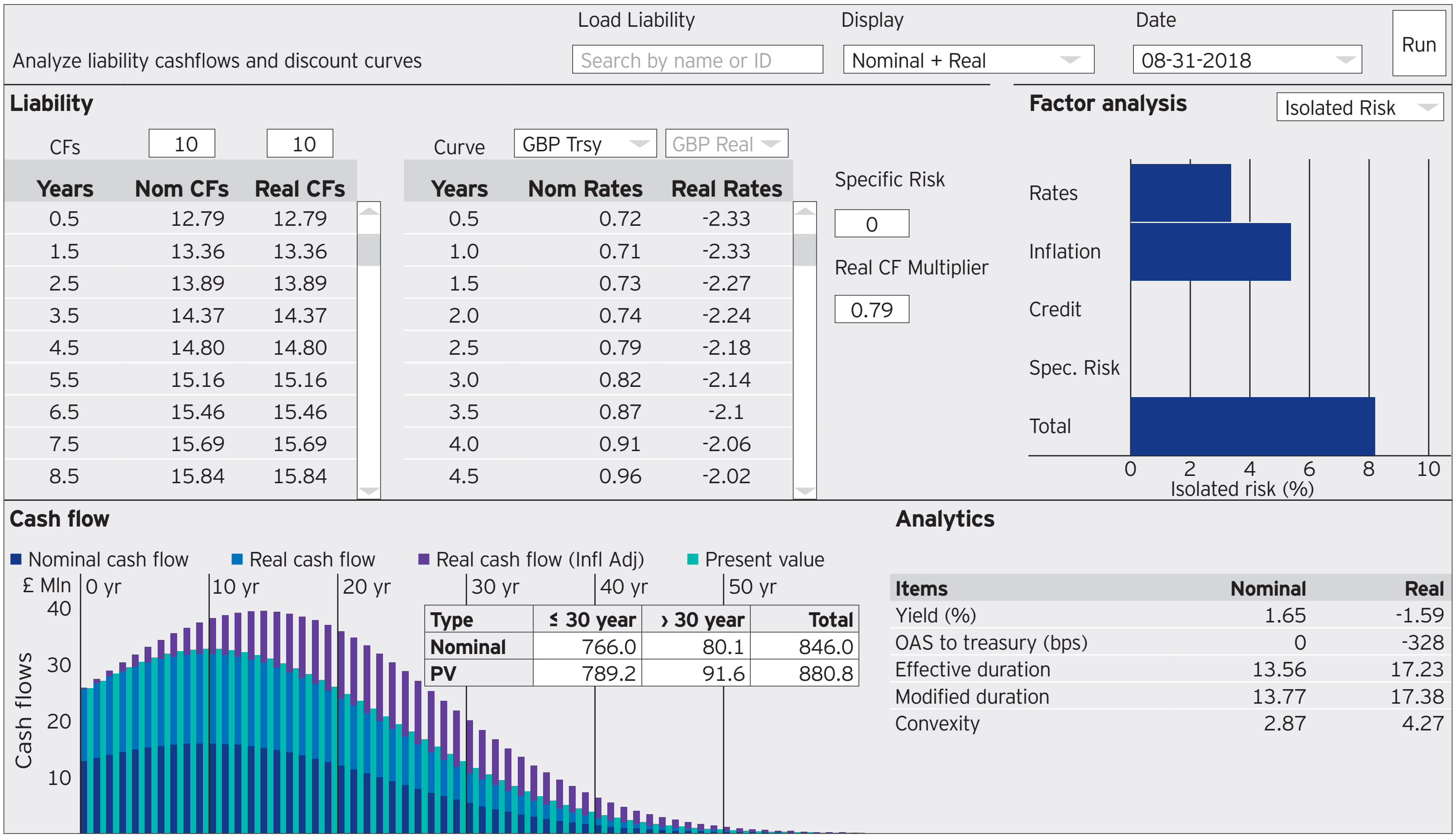
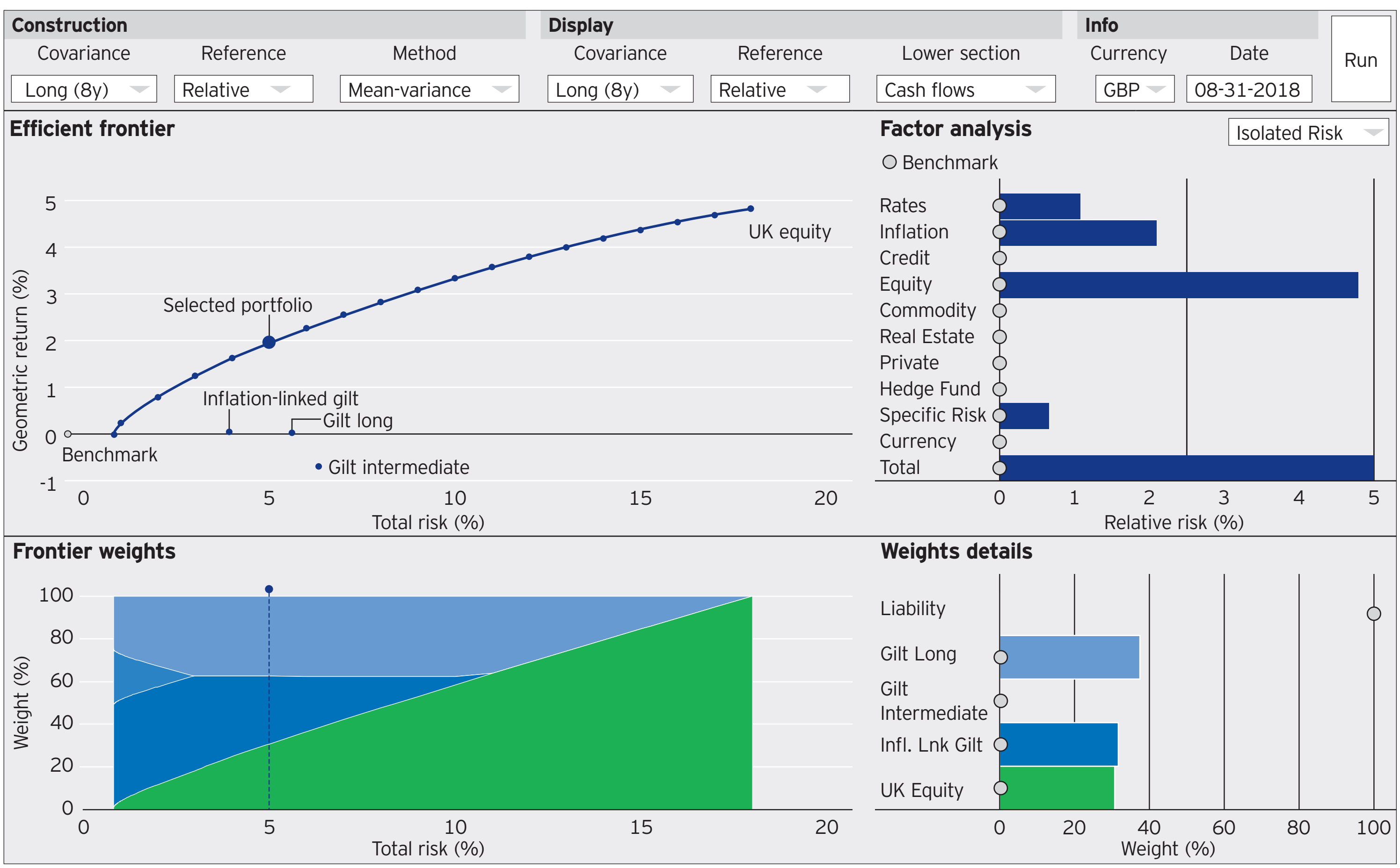



**Figure C6b: Liability-driven investing – UK defined benefit plan**
Liability relative efficient frontier



# Case 7: Cash flow driven investing

## Creating cash flow matched portfolios

In this example we examine how to create a buy-and-hold portfolio that can defease a pre-specified set of liabilities. We assume six cash flow payments that span 6 years. We also assume that the investible universe is comprised of the defined maturity (DM) corporate bond ETFs as well as some hypothetical, similarly structured Treasury blocks. Solving this problem requires the use of the cash flow optimizer.

In the upper panel of Figure C7a, we can see the efficient frontier in the context of this problem. By definition, the frontier is relative, while the axes demonstrate the key trade-offs: the portfolio relative cost versus the portfolio quality.* The first thing to notice is that the efficient frontier slopes downward, becoming cheaper as we reduce the overall rating. This is expected, as a lower quality portfolio will have a higher yield, being able to produce the desired cash flows at a lower cost. A second thing to notice is that the highest quality portfolio on the frontier has a cost that is marginally higher than the present value of the liabilities. This should also come as no surprise, as the cash flows generated by the investible blocks do not perfectly align with the liability cash flows. This leads to some drag as we re-invest any mismatches at the cash rate.

The lower panel of Figure C7a shows the composition of the frontier. All the way to the left with AAA quality is a portfolio comprised entirely of Treasuries. As one would expect, as we relax the rating constraint, moving to the right, the portfolio allocations gradually shift from all Treasuries to all maturity defined strategy. This allows the portfolio to become cheaper. Another thing to notice is the sequence by which blocks of various maturities are transitioned. The first Treasury blocks to be replaced by maturity defined corporates are the longest maturity ones. This is because the longest maturity blocks have the biggest impact on the overall cost of the portfolio.

In Figure C7b, we compare the asset cash flows to the required cash flows for the selected lowest rated portfolio. As expected, given the discrete nature of the investible blocks, the annual cash flows do not appear to be perfectly aligned. However, the cumulative cashflows, shown on the right of the figure, indicate that at no time was there a cash flow deficit. This is at the core of the optimization algorithm where we ensure that we never need to sell any assets to defease the liabilities.

Finally, for non-fixed income assets, any expected cash flows can also be included in the analysis and visualized in different colors. For example, if equities were included, their dividend yield would be part of the analysis. Furthermore, other more ambiguous income generating assets can also be added based on user specified income generation estimates.

* In this example, we discounted the cash flows using the Treasury curve to derive the portfolio relative cost.

**Figure C7a: Cash flow driven investing**
Cost versus quality frontier

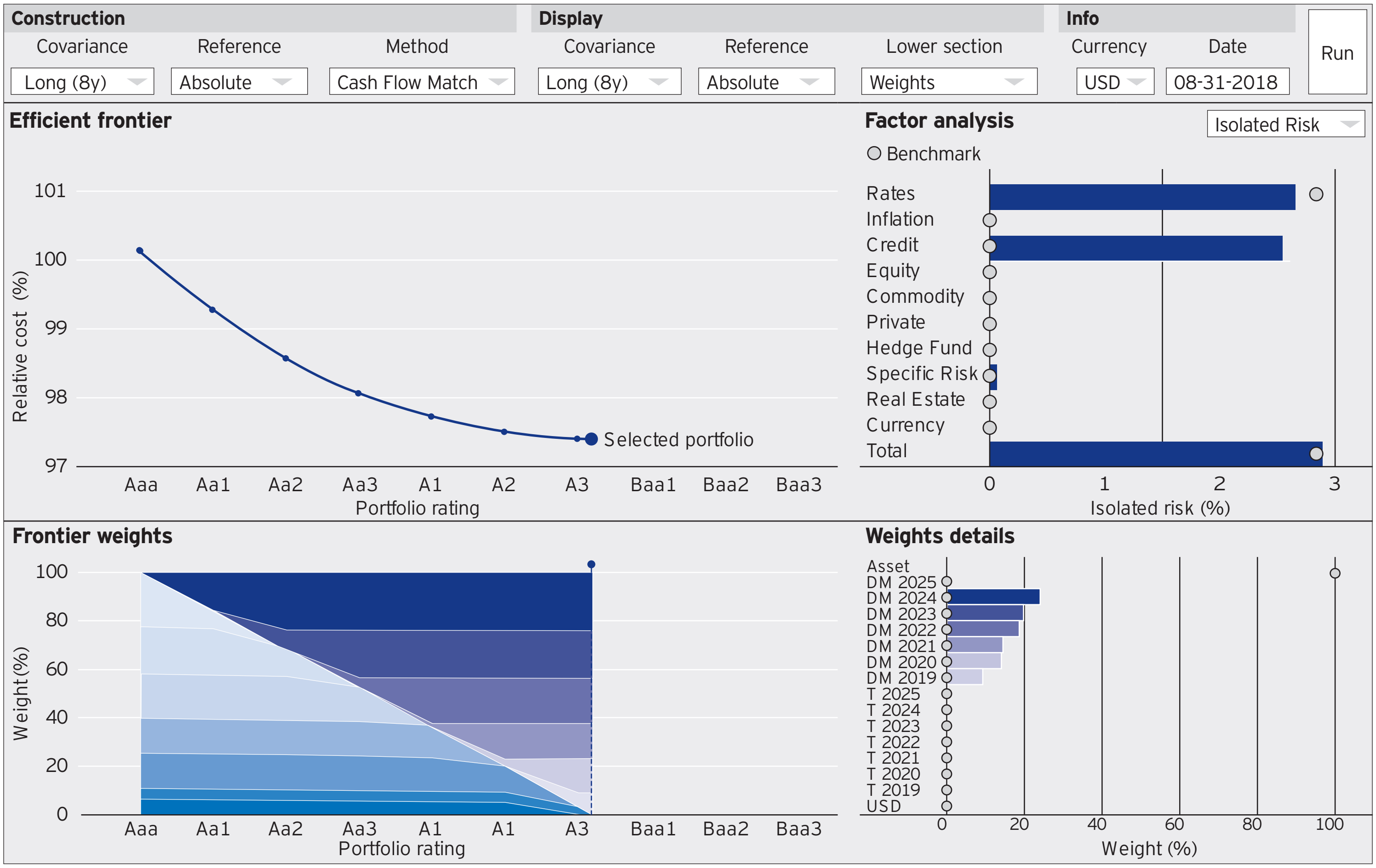
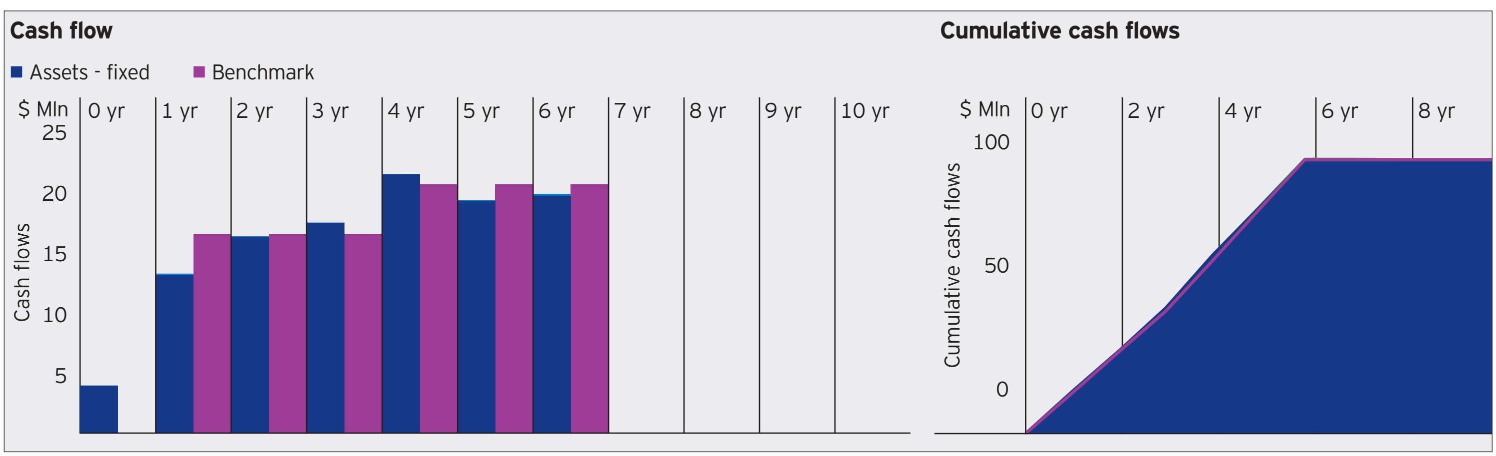



**Figure C7b: Cash flow driven investing**
Cash flow analytics



# Case 8: Portfolio construction with alternative assets

## Evaluating opportunities for improved risk-adjusted returns

In this case, we examine the impact of introducing alternative assets into a typical multi-asset portfolio. More specifically, we start by considering an efficient frontier comprised of a small set of fixed income and equity indices. We then consider a scenario where we allow for the inclusion of a handful of alternative asset classes, including Private Equity, Private Real Estate, Infrastructure, and Mezzanine Debt. This allows us to directly evaluate the impact of including alternatives. Also, given the complexities that are often involved with developing return, risk, and correlation estimates for illiquid asset classes, we have chosen to conduct the analysis using robust mean-variance optimization. This results in portfolio allocations that are more diversified and less susceptible to return estimate uncertainty. This also helps to avoid highly concentrated weights to alternatives that result as a function of their attractive risk to return characteristics. We have included a hypothetical portfolio that does not include alternative assets for comparison purposes.

In Figure C8a, the first thing to note is that the scenario efficient frontier that includes the alternative asset classes sits above the frontier of only traditional assets. This is to be expected as several of the alternative assets have higher expected returns as well as low correlations with the traditional assets. Second, when we examine the factor analysis for the selected efficient portfolio on the scenario frontier, we see that it provides the same return as the current portfolio with much better diversification across the macro factors. In this allocation, there is a very meaningful reduction in the equity risk that is replaced by direct real estate, pure private equity, and credit factor exposures. Finally, it is interesting to note that the exposure to alternatives is actually higher than public equities across the entire frontier. This could be a concern for an investor that has significant liquidity requirements. In such a situation, we could impose a constraint on the maximum allowable allocation to alternatives.

In Figure C8b, we present the underlying correlations for the asset universe considered. As can be seen, the correlation of alternatives with the other growth-oriented assets is lower than the correlations of the original growth assets among themselves. This is a direct result of the illiquid and idiosyncratic nature of these assets. It is interesting to note, however, that during stressed periods, such as during the Global Financial Crisis, the correlation of alternative assets with traditional assets does tend to increase. While this suggests that the benefits of alternatives may be somewhat overstated during stressed financial periods, there are benefits to investing in assets that provide access to a broad and differentiated set of return sources.

**Figure C8a: Portfolio construction with alternative assets**
Comparison of frontiers with alternative investments (scenario frontier) and without (frontier)

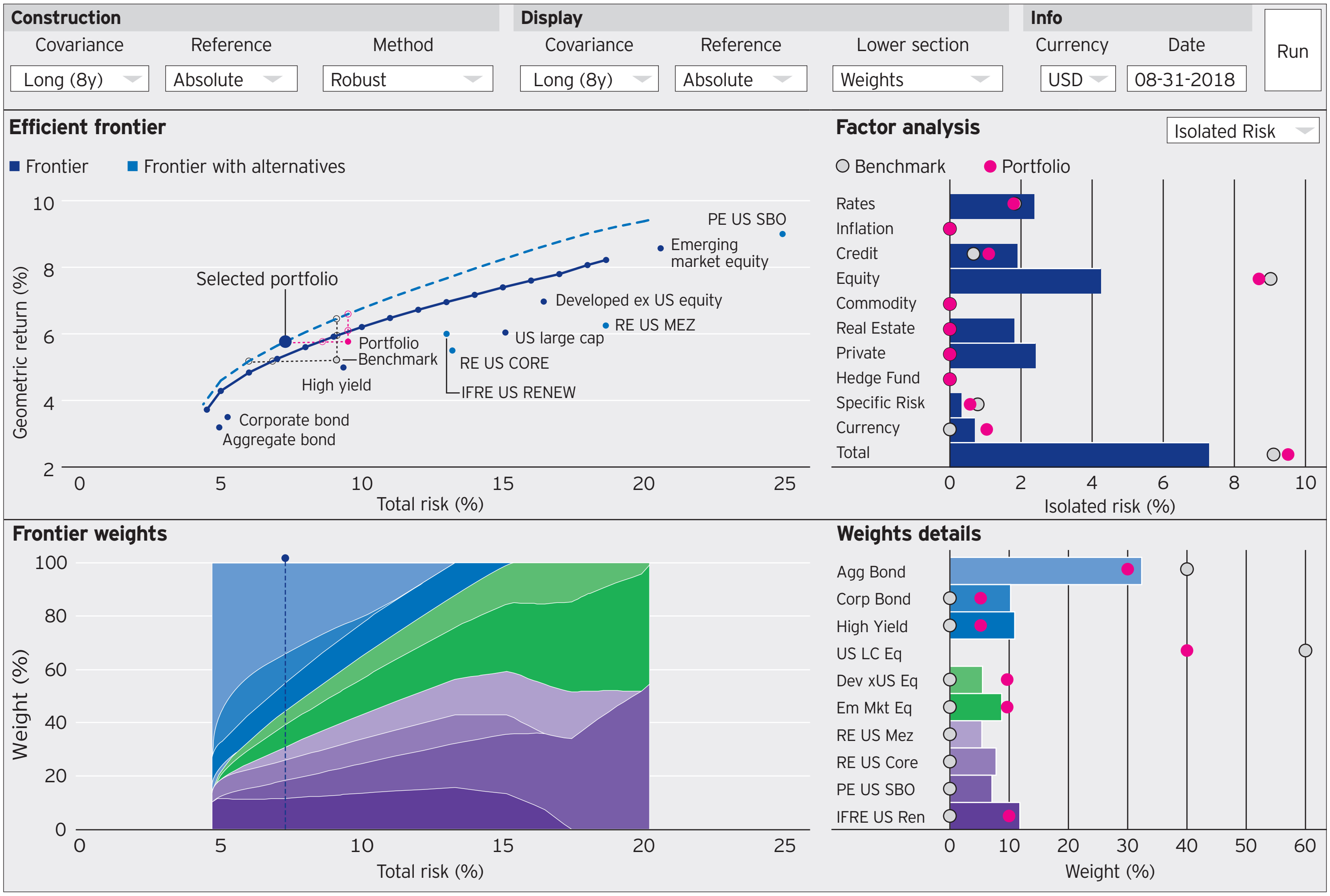
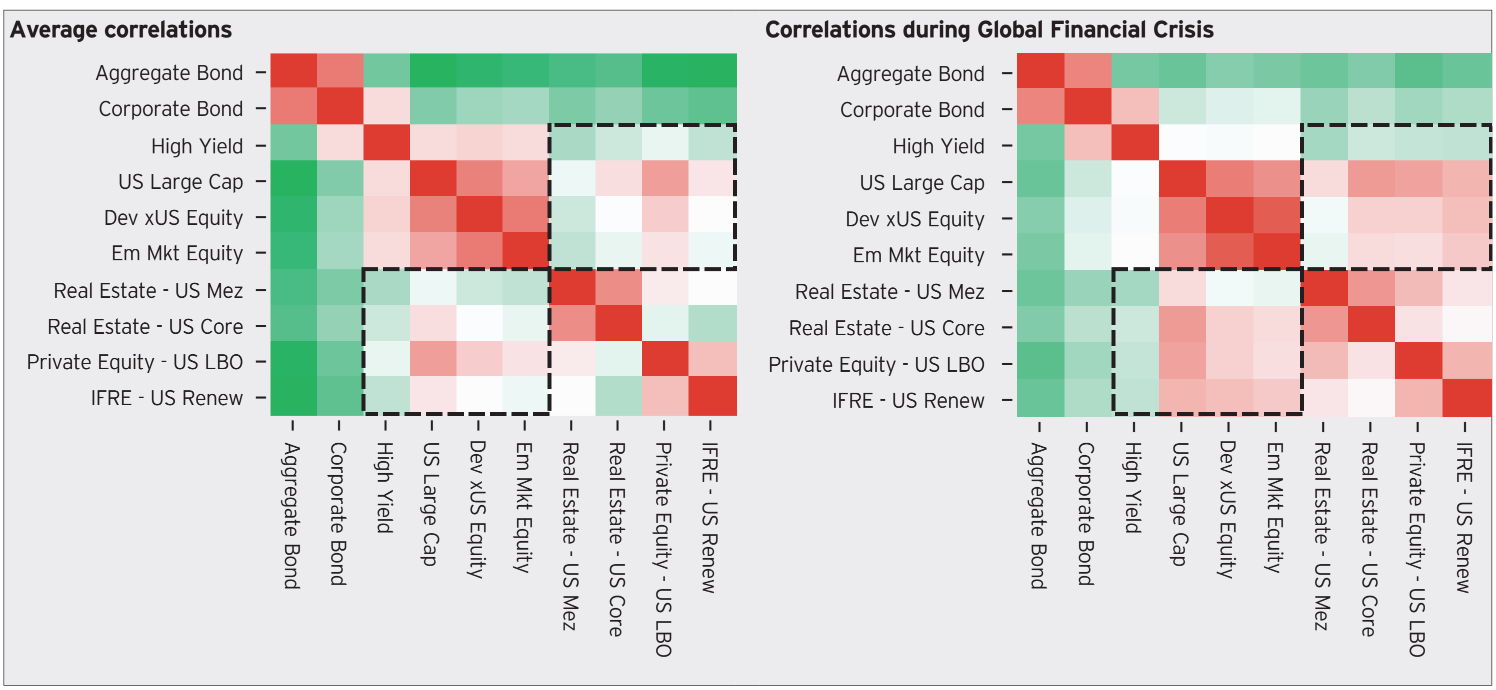



**Figure C8b: Correlations of traditional and alternative assets**
Comparison of alternative asset correlations on average and during the Global Financial Crisis



# Case 9: Portfolio evaluation

## Considering historical and hypothetical scenarios

As we construct outcome-oriented solutions, it is important to understand how such solutions might perform in various market conditions. Two types of analyses are particularly useful for such exercises. First, we look at how a portfolio would have performed during specific historical periods such as the GFC, Brexit, and the tech crash. Second, we look at how a portfolio would perform under various hypothetical market shocks, such as a 15% decline in global equities or a 100bps interest rate rise. For the purposes of this example, we consider a hypothetical efficient portfolio as compared to an existing portfolio and benchmark.*

Figure C9a focuses on the historical scenario analysis. Historical analyses rely on a set of specific time periods and their associated factor returns. Here, the current portfolio factor exposures are multiplied by the factor returns during the specific historical period so as to estimate what the return would have been during the respective period. It is important to recognize that the resulting performance represents the portfolio's performance given its current positioning and how it would have theoretically performed during such a historical period and not how it actually performed if it existed at the time. The figure demonstrates the selected portfolio's expected performance with blue bars as well as the original portfolio and benchmark expected performance with pink and gray dots, respectively.

As can be seen in the figure, the selected efficient frontier portfolio seems to do better than the original portfolio and the benchmark under almost all historical scenarios. The outperformance should come as no surprise as the portfolio was specifically optimized to have a lower risk. However, it is interesting to take a closer look at two cases where the selected portfolio did not do as well – 2004 and 2006 EM crises. To help understand why, in these cases the optimized portfolio was diversified away from US equities into a more balanced allocation with increased EM exposure. With emerging markets being at the heart of these historical periods, it should be expected that the optimized portfolios did not do as well as the original portfolios.

Figure C9b goes on to focus on the hypothetical scenario analysis. Here we have the option of performing the analysis in an uncorrelated or correlated mode. The difference between the two approaches is that in the uncorrelated mode only the shocked factors influence the performance of the portfolio, while in the correlated case the specified factor shocks are propagated to the remaining unspecified factors through their factor covariance structure. In this example, we demonstrated the correlated mode. Similar to the historical scenarios the performance of the selected portfolio as well as the original portfolio and the benchmark are shown. In almost every case, the optimized portfolio seems to perform better than both the original portfolio and the benchmark, as we would hope to see.

A particularly useful feature of both types of scenario analysis, shown in the lower part of each figure, is the ability to decompose the projected returns into underlying factor components. This allows us to more accurately determine the driving forces behind the observed performance and use this information to adjust the portfolio as may be desired. A subtle but important point to make here is that the analytics used to estimate a current portfolio's performance, either historically or in a hypothetical scenario, do not include second order pricing effects such as convexity or optionality.

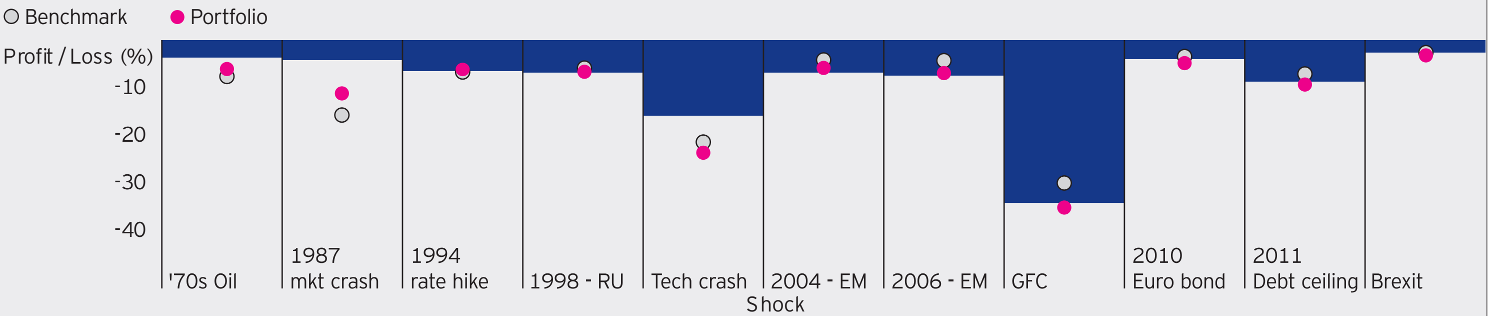
* In this example, we use portfolio information from Case 4 where we compare the Equal Return Portfolio to an existing portfolio and a benchmark.

## Figure C9a: Historical scenarios
Assuming uncorrelated sensitivities

**Scenario (excluding non-linear re-pricing effects)**

Historical

Uncorrelated




**Contributions to risk**

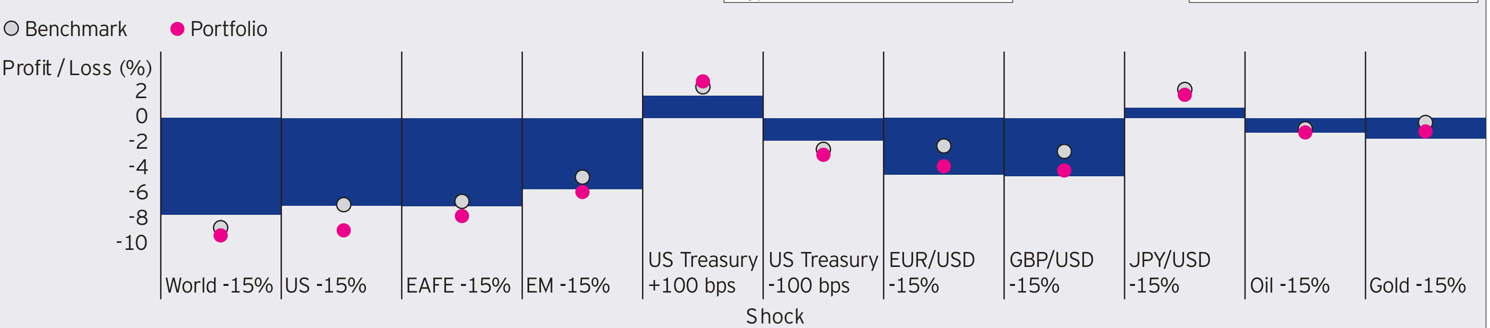
| Type | '70s Oil | 1987 Mkt crash | 1994 Rate hike | 1998 RU | Tech crash | 2004-EM | 2006-EM | GFC | 2010 Eurobond | 2011 Debt ceiling | Brexit |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Rates | -1.48 | 0.00 | -4.21 | 1.97 | 8.76 | -2.01 | 0.16 | 5.20 | 1.28 | 3.41 | 0.83 |
| Inflation | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Credit | 0.0 | 0.0 | -0.85 | -3.01 | -2.92 | -0.17 | -0.45 | -14.58 | -0.71 | -3.41 | -0.55 |
| Industry | -2.71 | -5.49 | -2.11 | -6.25 | -19.13 | -3.66 | -6.04 | -21.22 | -3.28 | -7.32 | -2.21 |
| Style | 0.05 | 0.04 | 0.34 | -0.42 | -0.47 | -0.14 | -0.30 | -0.61 | -0.09 | -0.19 | -0.08 |
| Commodity | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Real Estate | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Private Equity | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Hedge Fund | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Currency | 0.45 | 1.55 | 0.40 | 1.21 | -1.51 | -0.81 | -0.84 | -3.88 | -1.34 | -1.31 | -0.63 |
| X-Currency | 0.0 | -0.29 | -0.04 | -0.27 | -0.62 | 0.01 | 0.03 | 0.86 | 0.15 | 0.14 | 0.03 |
| **Total** | **-3.68** | **-4.18** | **-6.46** | **-6.77** | **-15.90** | **-6.79** | **-7.44** | **-34.22** | **-3.99** | **-8.68** | **-2.59** |

## Figure C9b: Hypothetical scenarios
Assuming correlated sensitivities

**Scenario (excluding non-linear re-pricing effects)**

Hypothetical

Correlated




**Contributions to risk**

| Type | World -15% | US -15% | EAFE -15% | EM -15% | US Treasury +100 bps | US Treasury -100 bps | EUR/USD -15% | GBP/USD -15% | JPY/USD -15% | Oil -15% | Gold -15% |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Rates | 0.64 | 0.64 | 0.47 | 0.32 | -3.20 | 3.20 | -0.45 | 0.22 | -1.27 | 0.20 | -0.36 |
| Inflation | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Credit | -1.60 | -1.49 | -1.38 | -1.07 | 1.77 | -1.77 | -0.49 | -1.02 | 0.82 | -0.35 | -0.17 |
| Industry | -5.79 | -5.42 | -5.01 | -4.05 | 3.30 | -3.30 | -1.48 | -2.07 | 2.24 | -0.78 | -0.44 |
| Style | -0.18 | -0.16 | -0.17 | -0.13 | 0.06 | -0.06 | -0.05 | -0.05 | 0.04 | -0.02 | -0.02 |
| Commodity | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Real Estate | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Private Equity | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Hedge Fund | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Currency | -0.70 | -0.47 | -0.87 | -0.66 | -0.14 | 0.14 | -2.03 | -1.65 | -1.04 | -0.19 | -0.64 |
| X-Currency | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| **Total** | **-7.64** | **-6.91** | **-6.96** | **-5.62** | **1.78** | **-1.78** | **-4.48** | **-4.57** | **0.80** | **-1.14** | **-1.63** |

# Case 10: Currency hedging
## Addressing currency risk

Invesco Vision has been designed to work with various base currencies and best represent the interests of clients across various economic regions. Whenever currencies are involved, two different issues need to be addressed. First, we need to be able to adjust return expectations to the relevant base currency, and second, we need to address any embedded currency risks. For the former, we rely on an interest rate risk parity model which implies that currencies will appreciate/ depreciate by the same amount as the respective sovereign interest rate differentials. For the latter we rely entirely on the risk model covariance matrix.

To demonstrate these ideas in practice, we start by constructing an example based entirely on USD-denominated asset blocks as shown in left portion of Figure C10a. Here we are viewing the assets from a USD perspective and there is no embedded currency risk. As can be seen, the efficient frontier begins at low risk levels, comprised almost entirely of the US Aggregate bonds and gradually adds other components such as equities and high yield to further increase the return.

We then move to the right portion of Figure C10a. Here we change the base currency to EUR. More specifically, we assume that a European based investor is looking at the same exact asset blocks as before and is using them to construct a EUR-based efficient frontier. There are several things to notice. First, the returns associated with all of the asset blocks and the efficient frontier itself are lower. This is a direct result of higher USD interest rates implying a future depreciation of the dollar vs the euro as dictated by the interest rate parity model. Second, all risk values are meaningfully higher. This is also to be expected as the portfolio now encompasses currency risk. Finally, the asset allocation structure along the efficient frontier is slightly different. This is due to the optimizer exploiting currency factor correlations to improve the risk-return trade-off.

Figure C10b proceeds with the same asset blocks, continuing to view the problem from a EUR-based perspective. However, in this case we assume that all assets are currency hedged to the EUR. There are a few things to notice. First, the expected returns remain unchanged at the lower levels we observed in the previous unhedged example. This is dictated by the way currency hedging works. More specifically, the modeled cost of currency hedging is based on the very interest rate differentials we used to project how we expect exchange rates to evolve. So, while we get to the return adjustment through a different path, the resulting adjustments are identical. The second and most interesting feature of the efficient frontier is that the underlying asset allocation is now identical to the original USD based asset allocation. Putting the two observations together, we can conclude that the impact of switching to a different base currency, combined with full currency hedging,only results in a vertical shift of the efficient frontier. The extent of the shift is dictated by the interest rate differential between the two currencies.

**Figure C10a: Currency hedging**
Unhedged USD- and EUR-based efficient frontiers

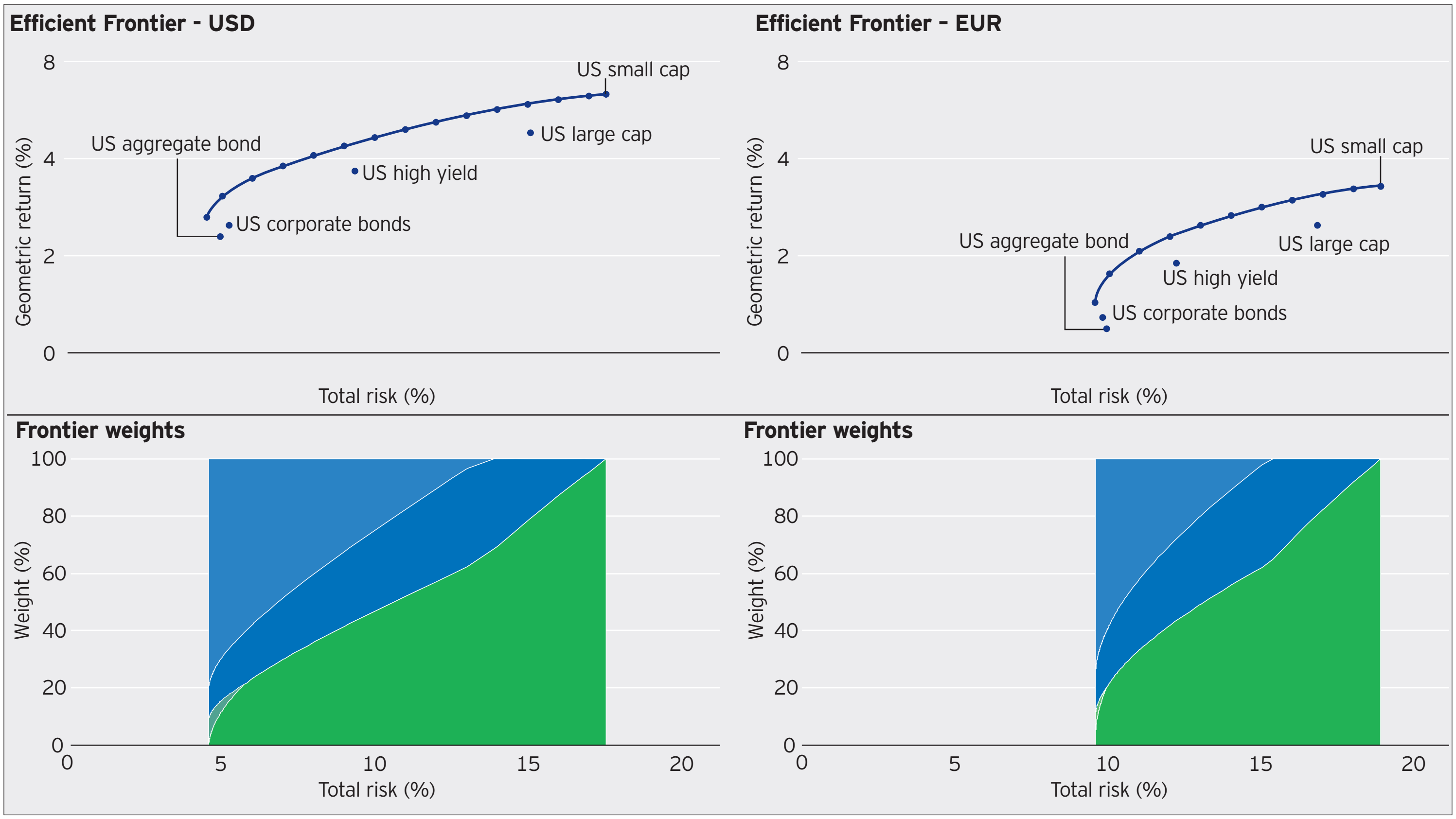
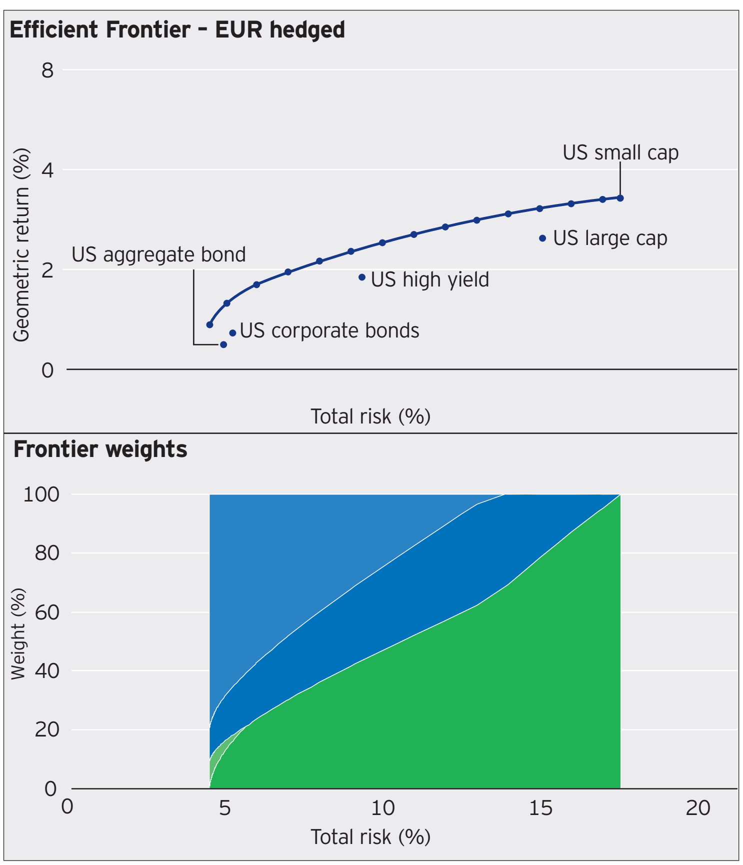



**Figure C10b: Currency hedging**
Hedged EUR based efficient frontier



# Case 11: Portfolio analysis with regulatory considerations
## Solvency II

Invesco Vision has been designed to address both economic risk focused challenges, as well as those presented by regulatory requirements. There is no better example for showcasing this capability than with an insurance entity. Here we examine a UK-based insurer that is subject to Solvency II. For reasons of simplicity, we do not include a liability in the exercise, even though this could readily be done. We assume the insurer is looking to invest in some basic asset classes such as sovereign bonds, corporate bonds, direct real estate, and equities.

In the top panel of Figure C11a, we showcase a Solvency II based efficient frontier for a set of selected benchmarks. We also include a hypothetical portfolio that is mostly comprised of Gilts and GBP corporate bonds with a small allocation to Sovereign EM, UK equities and UK property. As can be seen, the frontier looks very similar in shape and spirit to a typical economic efficient frontier, with the main difference being that risk is now measured in terms of solvency capital requirement (SCR) charges. To create the frontier, we employ the published Solvency II correlation matrices under both negative and positive interest rate shocks. In this case, since there is no liability benchmark, the positive interest rate shock will be used across the entire frontier as it leads to the highest SCR charges.

In the lower panel of Figure C11a, we showcase the Solvency II SCR decomposition for three of the indices. Here, the UK Corporate block has SCR charges which stem from both interest rate exposure and spread exposure. The interest rate exposure is computed based on re-pricing the index based on EIOPA prescribed shocks for the UK sovereign curve. The spread exposure is computed based on the underlying bond spread durations and ratings, as dictated by the governing bodies. The real estate block is only exposed to the private real estate charges of 25%. In this example, we have assumed an unlevered property holding so there is no increase in charges due to leverage. In the case of a typical direct real estate fund, this would unlikely be the case. Finally, we show the Sovereign EM block that is exposed to interest rate charges, spread charges and foreign exchange charges as the bonds are denominated in USD and the based currency in GBP.

In Figure C11b, we revert to the standard economic risk and return axes. Here we can see that, in this context, the corporate index is showing up with lower risk than the Gilt index as dictated by its lower duration and negative correlation between spreads and rates. The Sovereign EM block is also showing up with lower relative risk than what we saw in the SCR framework. This is driven by the lower implied risk due to currency exposure. Finally, it is interesting to note the difference between the economic efficient frontier and the Solvency II frontier when viewed through the economic risk lens. It is evident that the two frontiers are noticeably different. If we dig deeper into the accompanying allocations, we will notice that the SCR frontier generally avoids the available spread assets while the economic frontier seeks them, especially in the lower risk solutions. These types of trade-offs are typical for these kinds of problems and will ultimately come down to what is most important to the investor and the level of improvement beyond which only trivial changes are observed.

EIOPA = European Insurance and Occupational Pensions Authority

**Figure C11a: Efficient frontier - Solvency II**
Solvency capital requirement efficient frontier

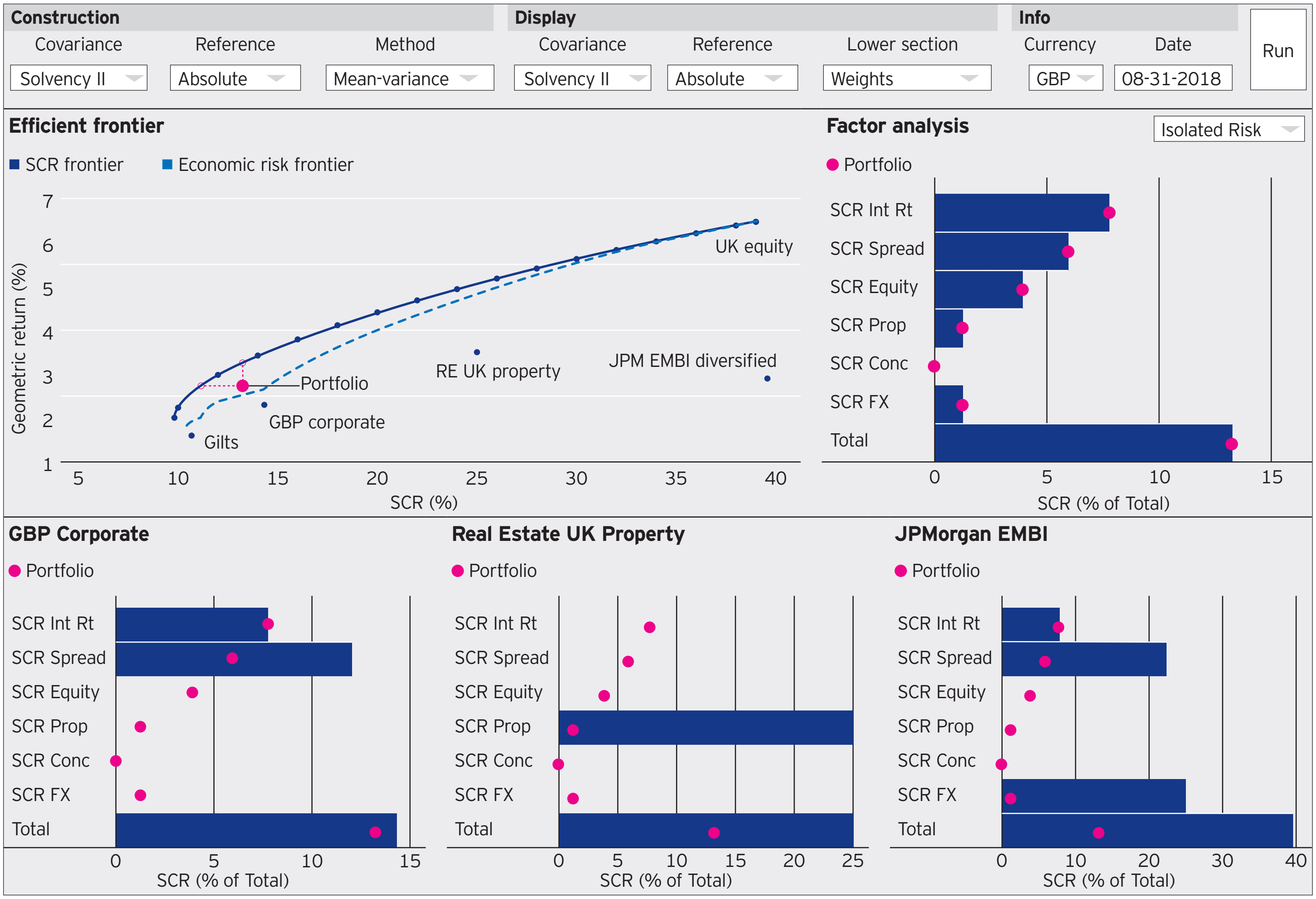
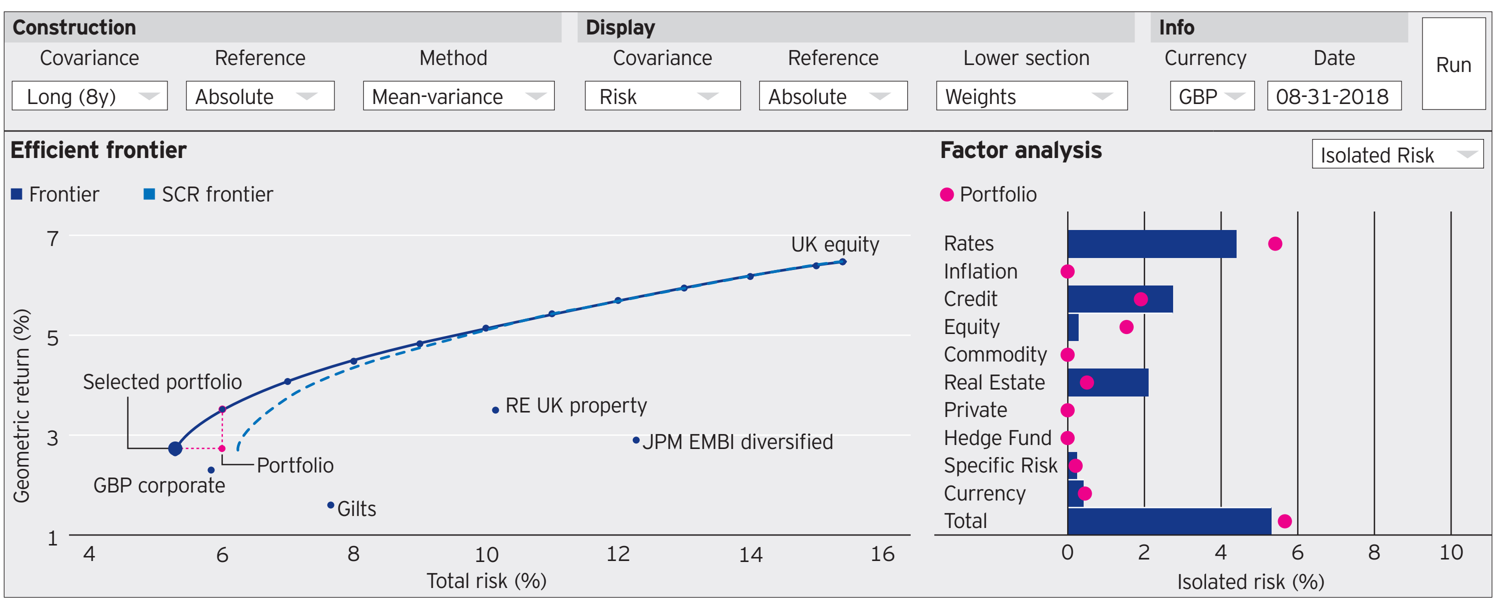



**Figure C11b: Efficient frontier - Economic risk**
Efficient frontier comparison



# Case 12: Portfolio analysis with regulatory considerations

US Risk-Based Capital (RBC)
National Association of Insurance Commissioners (NAIC)

In this example, we are evaluating a portfolio within the context of economic risk as well as US Risk-Based Capital (RBC) charges. To better understand the extent of efficiency for the portfolio, we are also creating efficient frontiers within both frameworks.

Looking at the upper panel of Figure C12a, an RBC-efficient, low-risk portfolio would consist almost entirely of agency mortgage backed securities. However, this is a very concentrated portfolio; intuitively, a more diversified asset allocation should be the goal. Supplementing the analysis with an economic efficient frontier will tell us what a more diversified portfolio could look like.

The lower panel in Figure C12a shows the result of an optimization based on economic risk. Here we see a more diversified allocation with meaningful investments in high yield bonds, emerging market debt and private equity. However, these asset classes entail significantly higher RBC requirements, and insurers have limited capital budgets for their investment portfolios. Considering this, along with the first figure, it is clear that trade-offs must be made to balance both economic and RBC perspectives.

Figure C12b plots a representative insurance portfolio that is commonly seen in practice. Note this portfolio falls between the economic and RBC frontiers, highlighting the trade-offs mentioned above that are typically made in the real world. But can this portfolio's efficiency be improved?

One important aspect of the RBC framework in the US is its punitive treatment of commingled fund investments. This is true even for high-quality bond funds. One could argue that a portfolio of investment grade bonds held via a mutual fund should be assigned the same capital charge as an identical portfolio held via a direct separate account, but absent a fund-level NAIC designation, the RBC framework assigns an equity capital charge to the bond mutual fund. This is one reason insurers strongly prefer direct separate account investments whenever possible. Figure C12b also shows the improvement in RBC efficiency when the initial portfolio's bond fund investment is moved to a direct separate account implementation.

**Figure C12a: Portfolio construction with regulatory considerations – Risk-based capital (RBC)**
Efficient frontiers based on RBC (top) and economic risk (bottom)

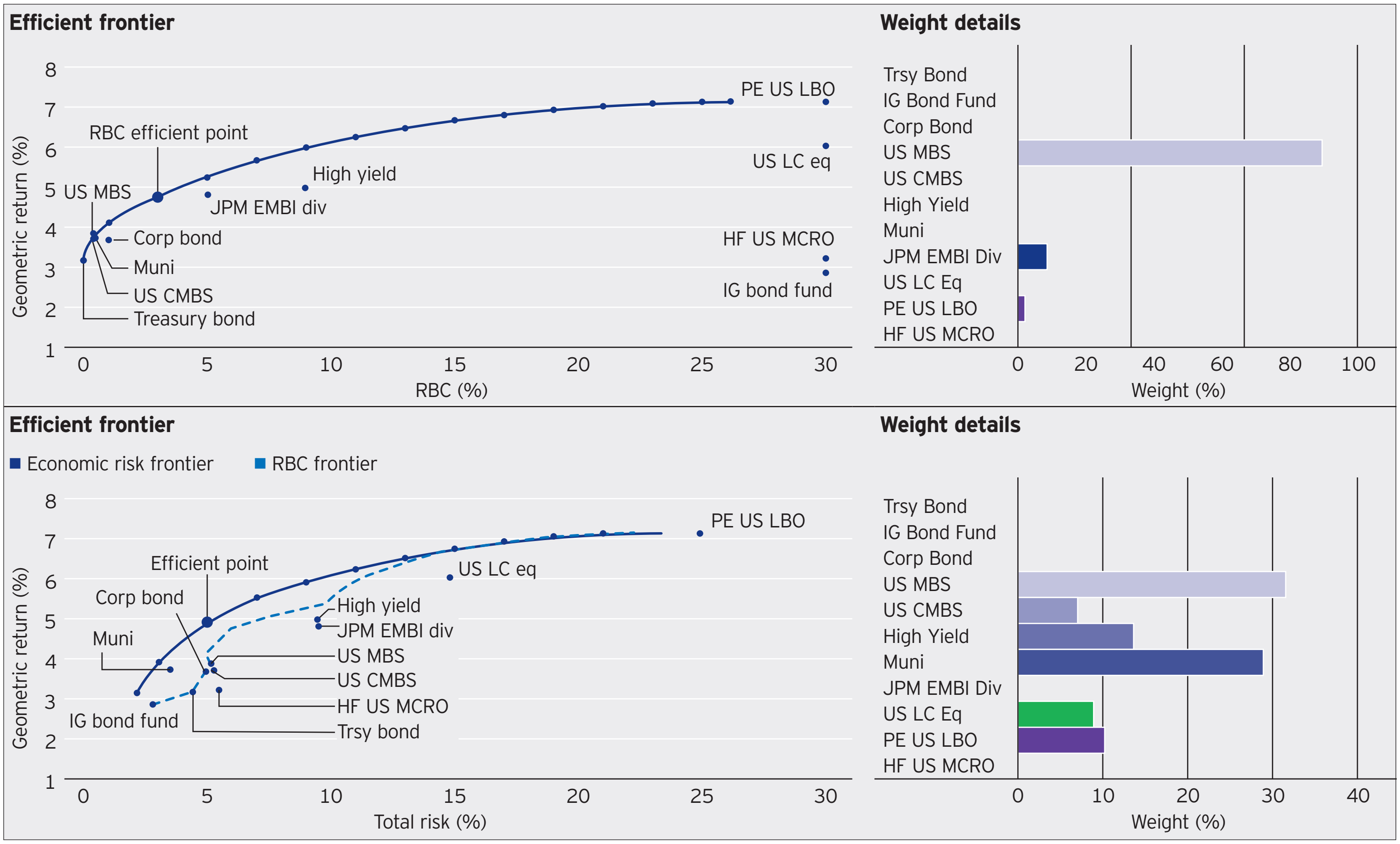
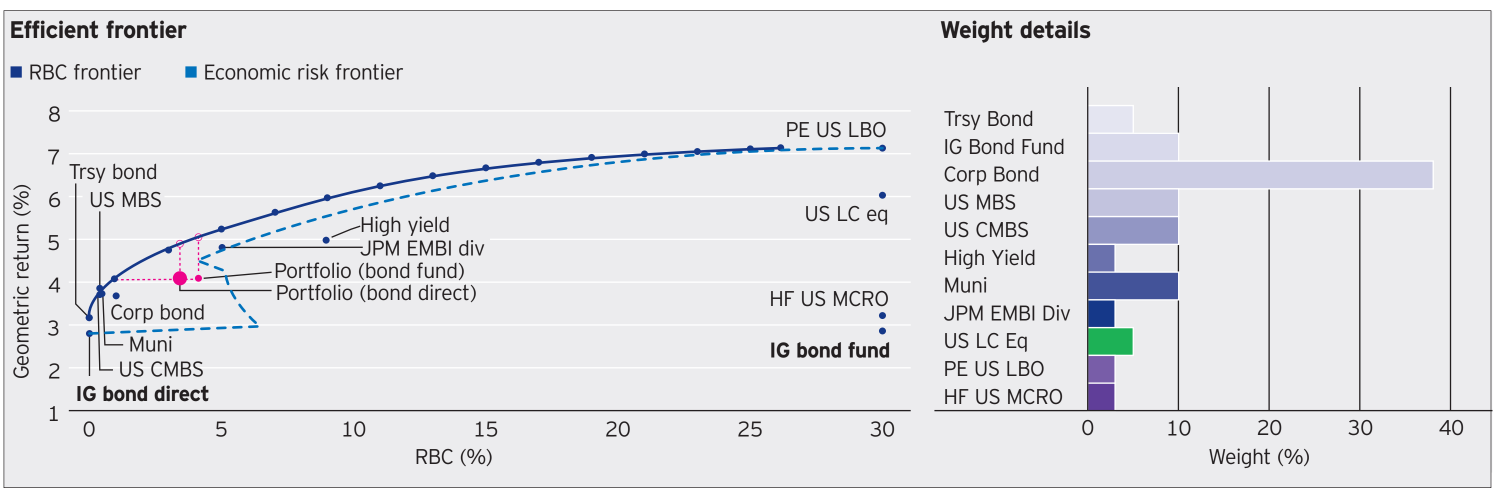



**Figure C12b: Hybrid RBC/economic risk portfolio**
RBC impact of fund-based versus direct investment – Investment grade bond



# Case 13: Model portfolio analytics
## Evaluating target-date funds

The use of model portfolios within retail and defined contribution plans is becoming prevalent. The approach offers pre-packaged, single-fund solutions that are constructed to help investors achieve their investing objectives. These solutions are frequently structured in the form of target date portfolios, where allocations become increasingly aggressive as target dates extend further into the future. In this case, we examine a set of target date fund series from 2020 to 2060. We discuss some key pitfalls to be cognizant of as well as some relevant practical capabilities provided by Invesco Vision that allow for easy comparisons of the different solutions.

In Figure C13a we present the five hypothetical target date portfolios. The blue dots indicate how these portfolios look when measured using expected arithmetic returns while the purple dots show the results in terms of geometric (compound) returns. First, we examine the portfolios in arithmetic return terms and notice how the shorter-term portfolios have lower total risk and lower expected return than their longer-term portfolio counterparts. We also see how the expected returns of the portfolios are rising at a rather linear pace as we move from shorter to longer dated solutions. We then examine the portfolios in geometric terms. Here, we notice that while the longer-dated portfolios show increasing levels of risk, they offer only marginal benefits in terms of expected return. This is driven by the volatility drag that is expected from these more aggressive solutions. In this context, it is unclear whether the 2060 portfolio should ever be preferred over the 2050 portfolio. The expected return pickup in geometric terms is a mere 19 bps while risk increased by over 120 bps.

In Figure C13b we present the underlying isolated factor risks, contributions to risk, along with fund asset class weights. As can be seen, the shorter dated portfolios have a heavy weight to credit and rates while the longer dated portfolios are increasingly reliant on equities. This can be seen in terms of both weights as well as in isolated risk. The contribution to risk makes this apparent shift even more transparent where equity drives more than 95% of the total portfolio risk for the 2060 portfolio. While this is not necessarily a bad thing, especially for the very long dated solutions, this level of transparency can help provide valuable insights regarding how the portfolio is expected to behave.

**Figure C13a: Model Portfolios**
Expected arithmetic and geometric returns

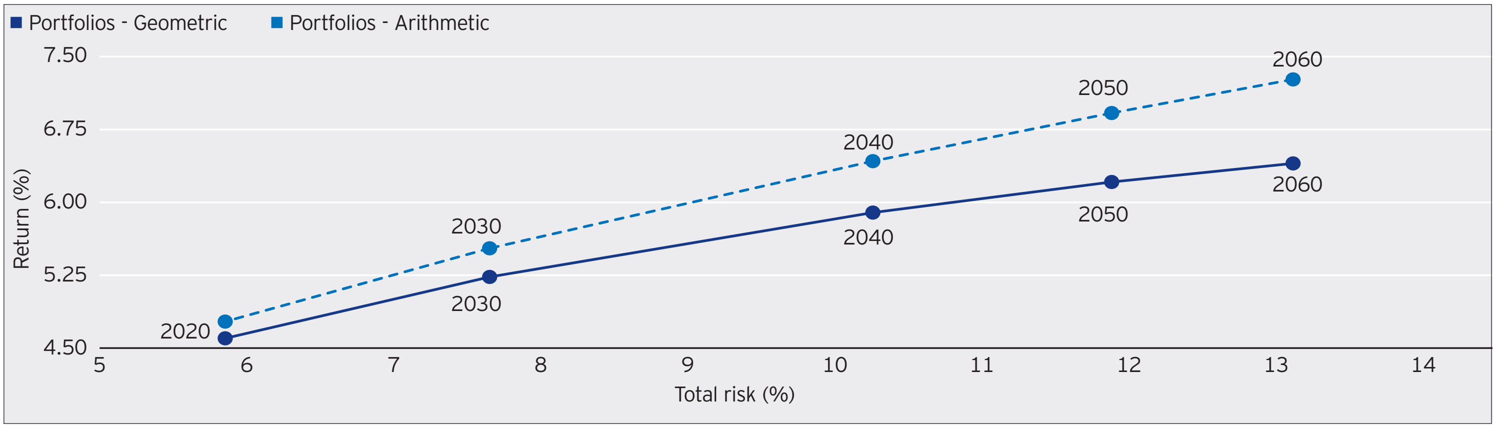
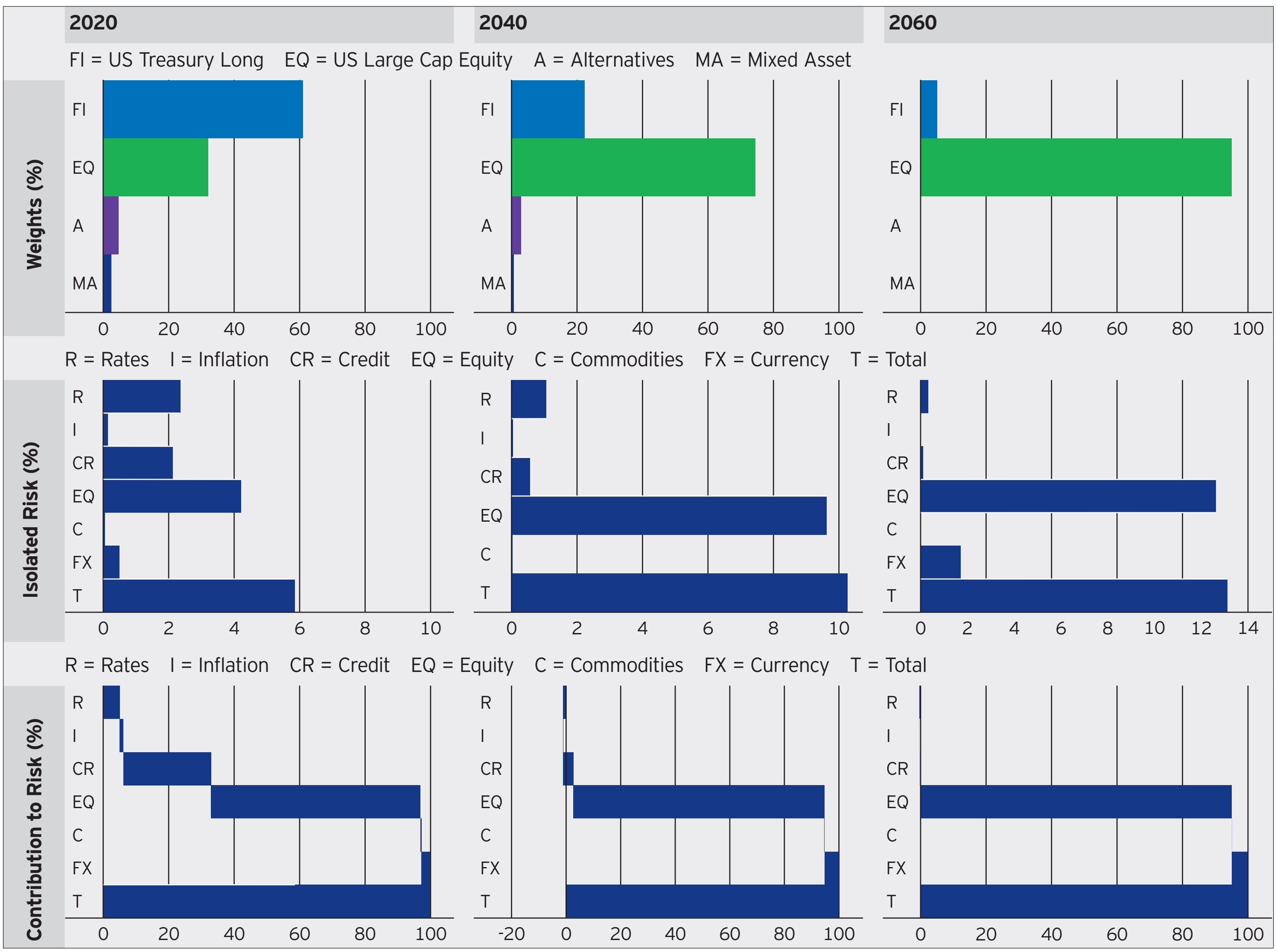



**Figure C13b: Portfolio characteristics**
Portfolio weights, isolated risk, and contribution to risk



# Case 14: Return agnostic solutions

## Portfolio construction without expected returns

While it is common practice to create portfolios through some form of mean-variance optimization, there are several other approaches that can be employed. Most of the approaches are aimed at eliminating the dependence on expected returns, which are the hardest optimization inputs to forecast correctly. In this example, we construct five such portfolios: equal weight, equal volatility, equal risk contribution, max diversification and global mean variance. To keep the example simple while trying to get exposure to a diverse set of factor exposures, we use three assets: US Large Cap (S&P 500), US Long Treasuries (Bloomberg Barclays Long Treasury) and Commodities (Goldman Sachs Commodity Index).

In Figure C14a, we chart the individual assets, the five return-agnostic portfolios, and the mean-variance efficient frontier with total risk (standard deviation) on the x-axis and geometric mean on the y-axis. What is immediately evident is that, while many of the return agnostic portfolios are near the mean-variance efficient frontier, the only portfolio that truly resides on the frontier is the global minimum variance portfolio. This reminds us that we can always construct an efficient portfolio that has the minimum risk without requiring any views on expected returns. The lower portion of Figure C14b summarizes the asset weights, isolated risk and contribution to risk for each of these solutions.

The simplest portfolio to create is the equal weight solution. In this case, the weights are simply derived by dividing 100% by the number of assets. An examination of the isolated risk and contribution to risk for this portfolio shows that even though the portfolio appears to be well diversified, commodities play an oversized role in the risk of the portfolio. This is a function of the amount of risk for each of the assets included. It should be noted that while the notion of this type of naïve diversification can be of interest, investors should evaluate the assets to be included carefully. Consider a three-asset equal weight portfolio where two assets are close substitutes, such as a S&P 500 ETF, a Russell 1000 ETF and a US Aggregate bond ETF. In such an instance, naïve diversification may actually lead to greater concentration and a higher exposure to risk than might be expected.

The equal risk, or risk-parity portfolio as it is sometimes called, is also a straightforward solution with the weights being set to be inversely proportional to the asset volatility. In this case, we see how the commodities sector gets the lowest weights, followed by stocks, and then bonds.

We then move to the portfolios whose weights can no longer be algebraically derived. First, we look at the equal risk contribution portfolio. This solution differs slightly from the equal risk solution, as it also considers asset correlations. As can be seen, the isolated risks are no longer the same, with commodities having the least isolated risk.

Moving to the maximum diversification solution which has the objective of maximizing the diversification ratio. This ratio is defined as the weighted average of the volatilities divided by the portfolio volatility. Here we see how stocks and commodities both have lower isolated risk and contribution to risk than fixed income.

Finally, we examine the global minimum variance portfolio. The objective here is to identify the portfolio with the lowest risk. This approach has recently received significant attention, specifically when looking at constructing portfolios within a specific asset class. The reason for the increased interest in this approach is that adacemic literature has put forward the idea that such a portfolio may not only exhibit lower risk, but it may also offer a premium. This may not necessarily be the case within a multi-asset context and in most cases. We treat this solution mostly as a reference point.

All of these portfolios can be viewed as reasonable options to consider when there is low confidence in the ability to effectively forecast expected returns and should be part of an investor's toolkit. They can also be useful as a reference for comparison for portfolios under consideration.

**Figure C14a: Return agnostic solutions with mean-variance efficient frontier**
Return and risk comparison

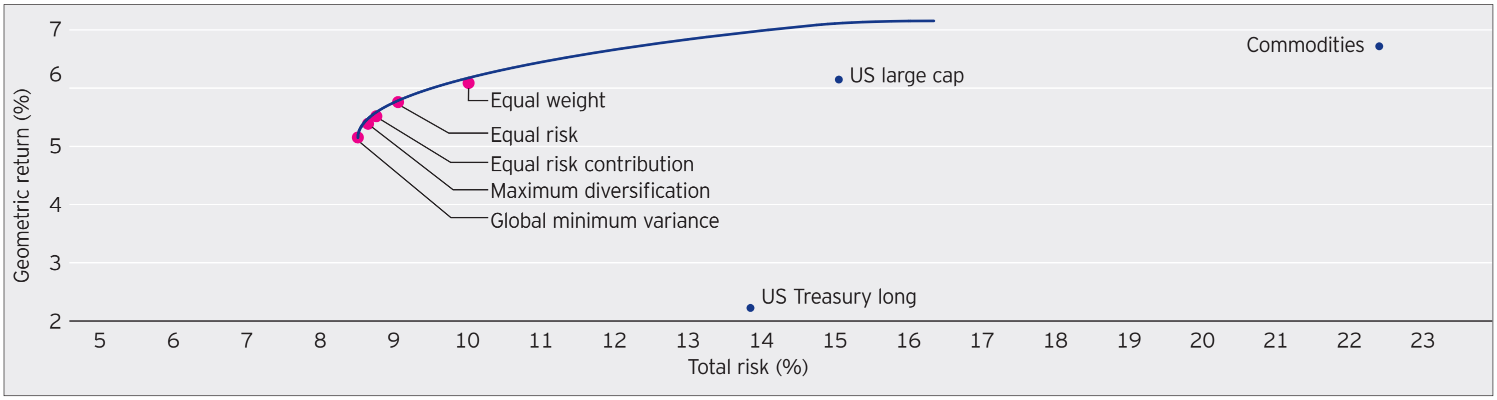
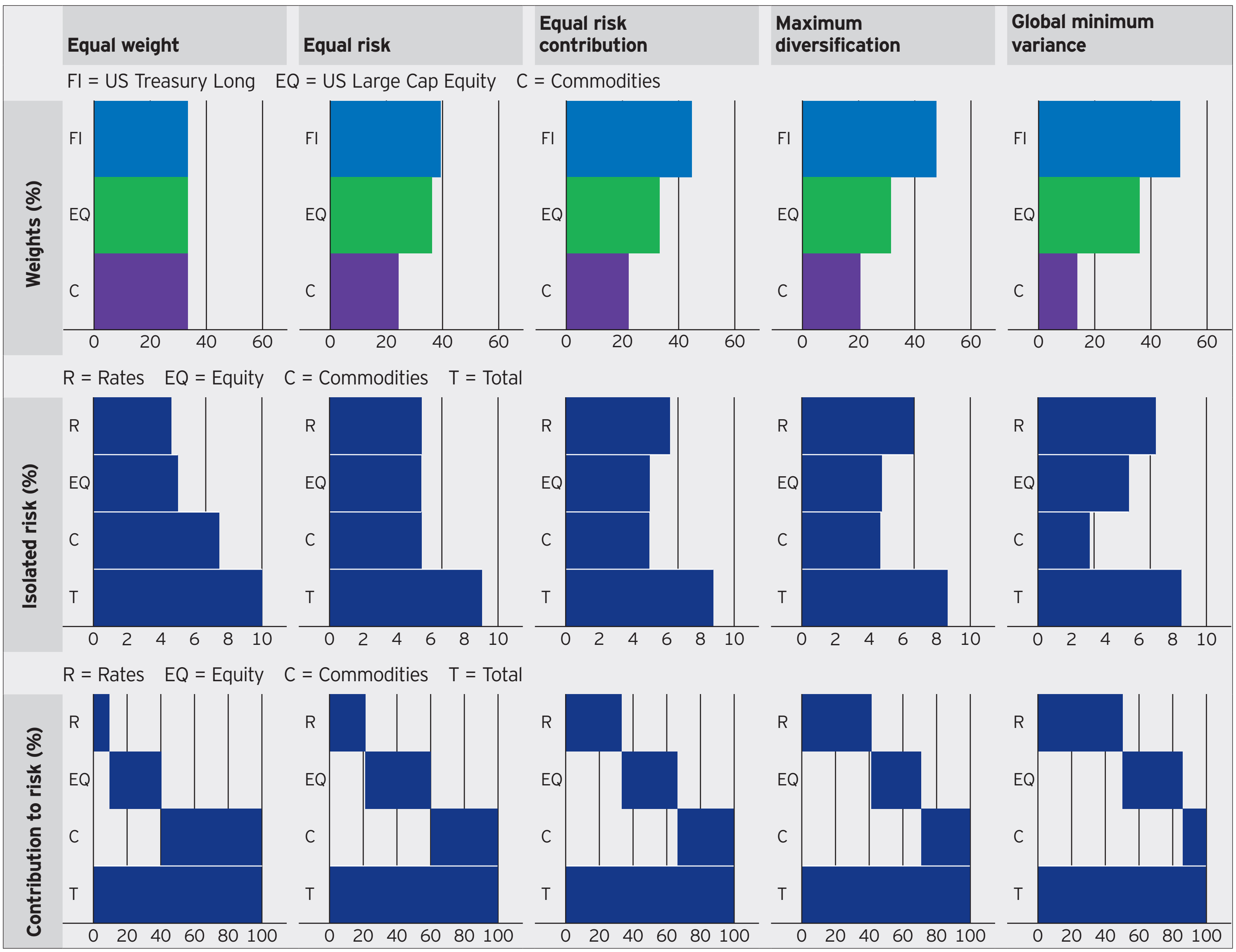



**Figure C14b: Portfolio Characteristics**
Portfolio weights, isolated risk, and contribution to risk



# Case 15: Multi-period optimization
## Creating portfolios to meet multi-period goals

The goal planning module allows users to create multi-period solutions. Unlike other analyses that entail a constant asset allocation over time, the goal planning module allows for time varying allocations. This is particularly important in the presence of expected inflows or outflows. Once the user has identified the investible universe, they can then define their investment horizon, along with any expected cash inflows and outflows, and select their risk measure of interest. It is important to note that in these types of problems, the distribution of terminal wealth outcomes may be meaningfully different from that of a normal or lognormal distribution, which makes the choice of risk measure particularly important.

Figure C15 presents a specific example where we assume starting capital of $1,000,000 with annual inflows of $300,000 per year for 10 years followed by $300,000 per year outflows for 10 years. The objective in this example is to maximize the expected wealth at the end of 20 years subject to the specified cash flows for various levels of terminal wealth uncertainty. On the asset side, our investment opportunity set includes US large cap equites, non-US developed market equities, US aggregate bonds and cash.

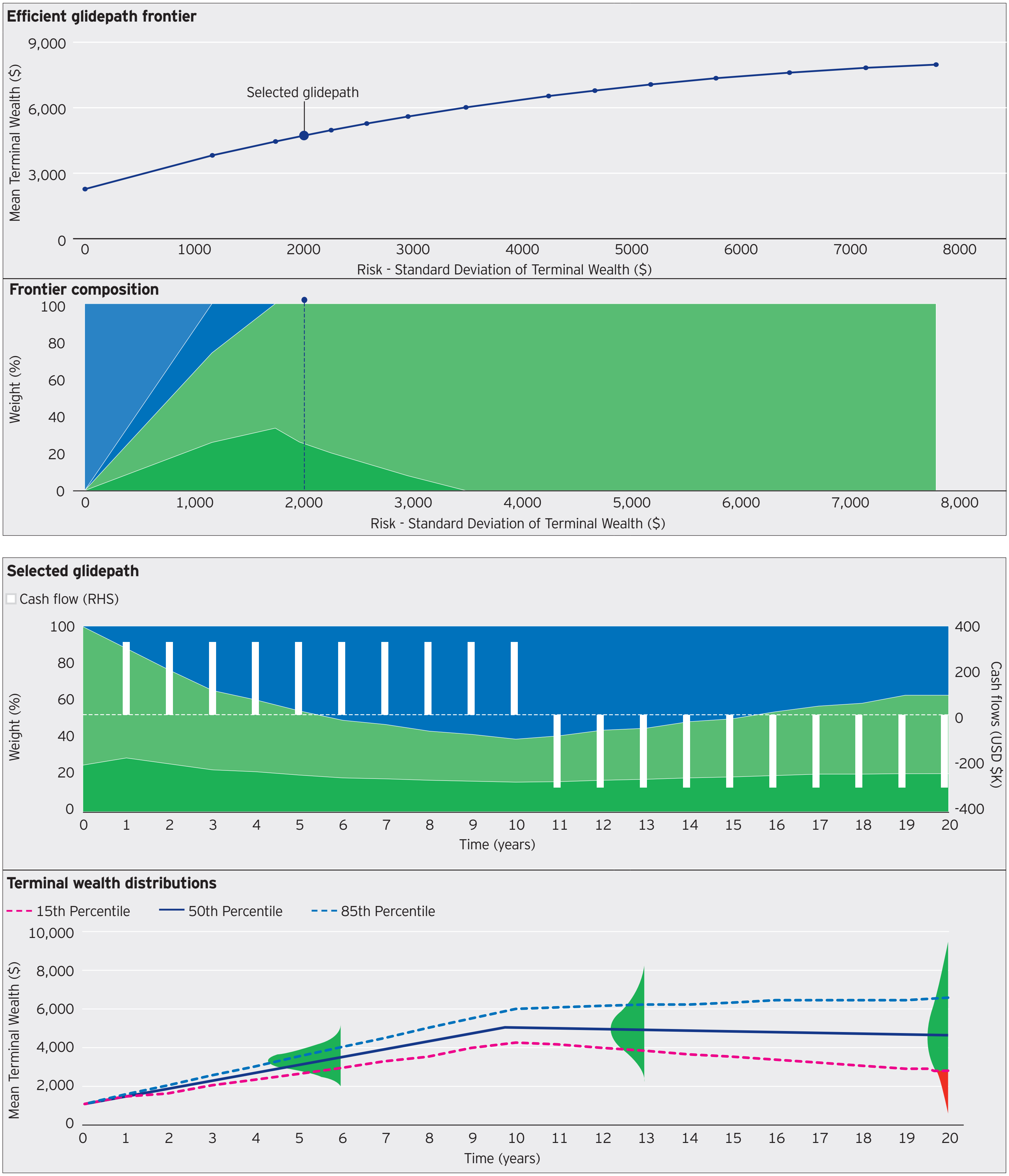
The top panel of Figure C15 demonstrates the resulting frontier in terms of expected terminal wealth and the accompanying volatility in terminal wealth. As we would expect, higher terminal wealth values are associated with more risky outcomes. The second panel shows the composition of the frontier at inception. This is the allocation that would be pursued at the start of the exercise. The third panel goes on to depict the glidepath associated with the selected point on the efficient frontier. This glidepath depicts how the asset allocation should evolve over time to best achieve the desired outcome.

As we examine the glidepath in more detail, there are several things to notice. First, at inception, the glidepath entails an allocation which is identical to the one shown in the second panel. The glidepath can effectively be thought of as a third dimension to the second panel. Second, and most importantly, we notice that the glidepath entails a period of de-risking prior to the cash outflows. This is very much in line with what we recommend as practitioners to an individual approaching retirement. The logic behind this gradual transition is that as we approach retirement and invested capital is increasing, risk needs to be reduced before outflows begin. The optimizer seeks to mitigate the impact of an instance where the retirement period begins with a significant market downturn. Something like this would prove very costly from a terminal wealth perspective. It is preferable to accept greater risk earlier in the investment horizon than at this critical point.

It is also important to notice that while the glidepath illustrates how the allocation is expected to evolve, the realized glidepath may evolve quite differently. Namely, the optimal asset allocations shown represent the average allocations to implement at any particular point in time. Optimal investment weights may follow another glidepath that is a function of actual portfolio value and market conditions through time.

Finally, in the bottom panel of Figure C15, we show the expected wealth distributions through time. The distributions graphically represent the likelihood of the portfolio's value as a function of time and provide intuition on the evolutionary nature of the investment problem.

**Figure C15: Multi-period optimization**



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