Portfolio construction

Absolute risk optimization

Given a vector of expected annualized arithmetic returns μ that are determined through our capital market assumption process, and an annualized asset level covariance matrix Σ_A that is derived from either our economic, Solvency II, or NAIC risk model, our basic goal in absolute risk optimization is to minimize the absolute portfolio risk for any given portfolio return.

Mathematically, the problem is to find the best collection of investment (or portfolio) weights W* that satisfies:

$$w^* = \underset{w}{argmin}\ \ w^T \Sigma_A w, \text{ such that } \ w^T \mu = \mu^*$$

for a collection of values $\mu_{min} \leq \mu^* \leq \mu_{max}$. Additionally, we may simultaneously require that each portfolio weight fall within a range of acceptable values, i.e., $a_j \leq w_j \leq b_j$ for $j=1, \cdots, N$ and also require that sum of the portfolio weights meet a required budget, i.e., $\sum_{j=1}^{N} w_j = B$. Typically, we take the value of the budget to be 1 meaning that the portfolio weights add to 100%.

Relative risk optimization

Relative risk optimization is a very similar problem to absolute risk optimization, only that in the relative risk case we seek to minimize risk relative to a benchmark or reference asset for any given portfolio return. We treat this problem as being long the portfolio and short the benchmark. Mathematically, the problem may be written as:

$$W^* = \underset{W}{argmin}\ \ W^T \Sigma_{A,\ relative} W, \text{ such that } W^T \delta\mu = \mu^*$$

where the optimal weight vector is now of the form:

$$W = (-1, w_1, w_2, \cdots, w_N)$$

and the return vector is express as being relative to the benchmark:

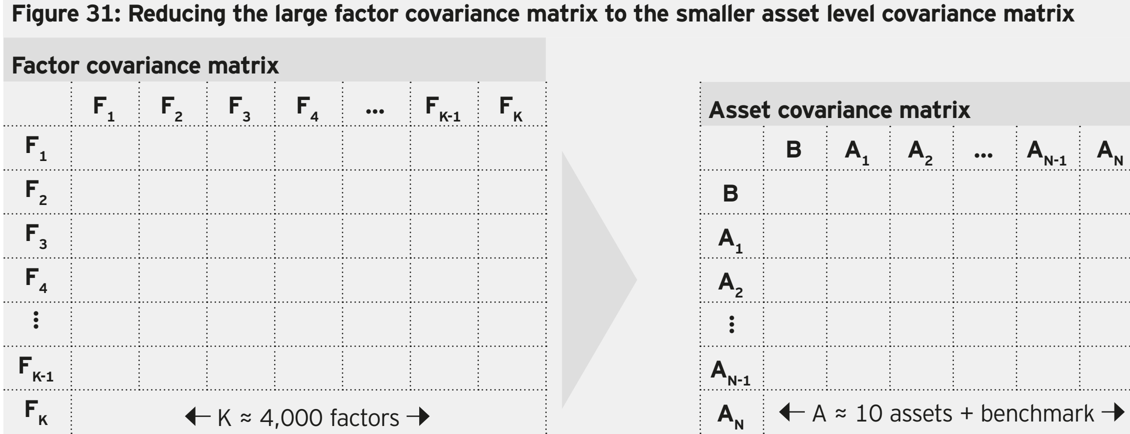
$$\delta\mu \ = \left(0, \mu_1 - \mu_b, \mu_2 - \mu_b, \cdots, \mu_N - \mu_b\right)$$

As in the absolute risk optimization problem, the "μ" is the absolute return for the portfolio and μ_b is the return of the benchmark of interest. Similar to how we converted the factor covariance matrix into an asset level covariance matrix (see page 17) we create the relative covariance matrix as shown below:

$$\Sigma_{A,relative} = \left(\beta_b, \beta_1, \beta_2, \cdots, \beta_N\right)^T \Sigma_f \left(\beta_b, \beta_1, \beta_2, \cdots, \beta_N\right) + \Lambda_{rel}$$

where Σ_f is the factor covariance matrix, i.e., the matrix of covariances between the risk model's factors and Λ_{rel} is the relative specific risk matrix. In Figure 31 we illustrate the translation of the nearly 4,000 by 4,000 factor covariance matrix to the benchmark relative asset covariance matrix.

Figure 31: Reducing the large factor covariance matrix to the smaller asset level covariance matrix



Source: Invesco, BarraOne.

Robust mean-variance portfolio optimization
A mean-variance optimal portfolio is one that minimizes risk for any given portfolio return. However, unconstrained mean-variance optimization (MVO) can exhibit characteristics that are undesirable for some investors. For example, if some of the assets are close substitutes, asset weights can be unstable, i.e., a small change to the expected returns yields substantially different asset weights even though the distribution of returns provided by the portfolio is likely to be only marginally affected.

Additionally, unconstrained MVO can result in portfolio allocations that are highly concentrated in a single asset or across a small number of assets. In this sense, the basic implementation of MVO is not robust to the likelihood of errors in the estimation of expected returns and exposes investors to the possibility of overweighting underperforming assets. For at least these reasons it may be desirable, if not necessary, to modify the basic MVO framework.

Ceria and Stubbs (2006) have carefully considered the fundamental issues addressed above and have reformulated the MVO problem. At the heart of their modified portfolio optimization process, which they call Robust Mean-Variance Optimization, is the further incorporation of the uncertainty of the expected, or forecasted, returns. They start by assuming that the actual returns the portfolio will realize reside within an uncertainty ellipsoid of known size surrounding the expected returns. This is formulated as follows and is also visually depicted in Figure 32.

$$\left(\mu_{realized} - \mu_{expected}\right)^T \tilde{\Sigma}^{-1} \left(\mu_{realized} - \mu_{expected}\right) \leq \kappa^2$$

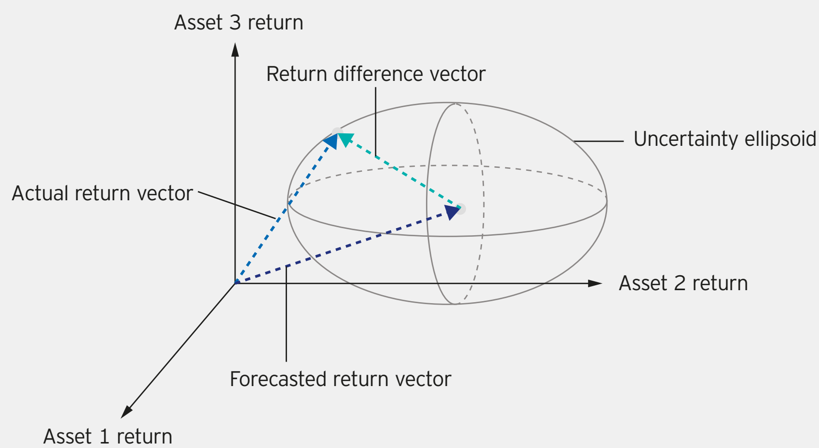
Where $\kappa^2 = x_n^2(1-\alpha)$ and x_n^2 is the inverse cumulative distribution function of the Chi-squared distribution with n degrees of freedom. Finally, $\tilde{\Sigma}$ is the uncertainty covariance matrix, not to be confused with the asset return covariance matrix, Σ_A.

Using the above, the Robust MVO problem can be formulated as follows:

$$w^* = \underset{w}{argmin} \ \ w^T \ \Sigma_A \ w$$
$$\text{such that } \ w^T \mu_{expected} - \kappa \sqrt{w^T \tilde{\Sigma} w} = \mu^*$$

for a range of candidate values $\mu_{min} \leq \mu^* \leq \mu_{max}$ and require that the asset weights satisfy lower and upper bounds and add to a portfolio weight budget of 100%. In this s etting, a penalization term (the second term in the above return constraint involving the square root) has been added to the return target constraint. By incorporating a relatively simple penalty term, the portfolio optimization process can account for expected return uncertainties, becoming less sensitive to small changes in forecasts, and providing optimal asset allocations that are more diversified than those provided by unconstrained MVO.

Figure 32: Uncertainty ellipsoid showing the distance between the actual and forecasted returns



Source: Invesco.

Return agnostic solutions

Given a set of assets, an investor may be interested in identifying possible allocations that are not reliant on return forecasts such as an equal weighted portfolio, an equal volatility portfolio, an equal risk contribution portfolio, a maximum diversification portfolio, or a global minimum variance portfolio.

Each of the above solutions require varying amounts of forecasted information ranging from no information (equal weight), to asset risk estimates (equal volatility), and finally, covariance estimates (equal risk contribution, maximum diversification, and global minimum variance). Figure 33 summarizes the data requirements and the corresponding mathematical formulations for each of the return agnostic solutions.

Figure 33: Inputs required for various portfolio construction methods

Construction method	Volatility forecast	Correlation forecast	Return forecast	Mathematical formula
Equal weight	–	–	–	$w_i^* = 1/N$
Equal risk	✓	–	–	$w_i^* = 1/\sigma_i$
Equal risk contribution	✓	✓	–	$w^* = \underset{w}{\operatorname{argmin}} \sum_{i,j=1}^{N} \left\| \frac{w_i(\Sigma w)_i}{\sqrt{w^T \Sigma w}} - \frac{w_j(\Sigma w)_j}{\sqrt{w^T \Sigma w}} \right\|$
Maximum diversification	✓	✓	–	$w^* = \underset{w}{\operatorname{argmin}} \frac{w^T \sigma}{w^T \Sigma w}$
Global minimum variance	✓	✓	–	$w^* = \underset{w}{\operatorname{argmin}}\ w^T \Sigma w$
Mean-variance	✓	✓	✓	$w^* = \underset{w}{\operatorname{argmin}}\ w^T \Sigma w$ such that $w^T \mu = \mu^*$

Source: Invesco, BarraOne.

Cash flow (liability) matching

An institution or portfolio manager that is expected to make a sequence of future cash payments is faced with a standard liability matching problem. In such a problem there is a well-defined schedule of future cash payments that must be made using the principle and coupon payments from a collection of fixed income investments. For this type of a problem, we seek to create a portfolio, subject to various constraints, that defeases the liabilities at the lowest cost possible.

Cash flow matching problems are rather straightforward in that they rely on linear optimization techniques. Given the universe of available fixed income securities, there is considerable latitude in choosing the subset of the available fixed income securities needed to meet the investment goal. Higher returns are often associated with lower credit ratings, and so a credit rating representing the average (or collective rating) of all of the fixed income securities held in the investment portfolio may be part of the optimization (i.e., minimize cost subject to some desired level of credit quality). Mathematically, we seek to minimize the following objective function:

$$x^* = \underset{x}{\operatorname{argmin}}\ p^T x$$

where x represents the vector of amounts of each of the possible fixed income securities to purchase, and p denotes their respective prices. The rest of the problem concerns the formulation of constraints.

Foremost, we must be able to make each anticipated liability payment at the scheduled times $t_1, t_2, \cdots t_K$. Additionally, we may be required to hold a minimum amount of certain fixed income securities such as government bonds and may similarly be required to hold no more than a prescribed amount of higher risk investments such as BBB-rated bonds. This means each investment must be constrained by lower and upper bounds. We also wish to maintain a positive investment balance at each point in time. This means that cash inflows from coupons, repayments, and previous reinvestments should be sufficient to meet cash outflows (liability payments) at any point in time. Lastly, the average or collective credit rating of the portfolio may be required to meet or exceed some pre-specified minimum rating. All of these portfolio attributes can be defined as linear equality or inequality constraints.

First, to meet the liability payments at each time period $t_1, t_2, \cdots t_K$, we impose the following equality constraint

$$(C \quad R)\binom{x}{b} = L$$

where C is a matrix that represents the cash repayments at each point in time for each security. R is a matrix whose entries represent the reinvestment opportunities for positive investment balances through time which takes on the form shown below, b represents the balance of the investment portfolio at each point in time, and L represents the stream of liability payments.

$$R = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 1+r_1 & -1 & 0 & & 0 & 0 \\ 0 & 1+r_2 & -1 & \ddots & 0 & 0 \\ & \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1+r_{T-1} & -1 \end{bmatrix}$$

Second, we require the investment balances all be positive so that b ≥ 0 for each time period. Third, to meet the requirements of minimum allocations and/or maximum limits to specific instruments, we generically require each asset to satisfy explicit lower and upper bounds. Finally, we impose a minimum portfolio-level credit rating constraint by converting the standard alpha-numeric credit ratings into numeric values.

Multi-period portfolio construction

Investors with a long-term investment horizon, during which there may be numerous cash inflows and outflows, face a multi-period portfolio construction problem. In particular, they must consider the totality of the cash flows and their ultimate financial goal in order to determine the best investment strategy through time. Such an optimal investment strategy results in a glidepath or a sequence of time dependent optimal portfolios in which to invest. The strategy can be used to address multiple objectives such as maximizing expected wealth at the end of a 30-year investment horizon subject to various inflows and outflows while not exceeding a specific level of uncertainty.

Modeling such a complex process goes beyond single-period mean-variance optimization and requires one to revisit the foundations of investing in uncertain markets. At its core, the modeling of portfolios in dynamic markets is an exercise in stochastic analysis. For $t=0, 1, 2, \ldots, T$, and cash flows $C = C_1, C_2, \ldots, C_T$, the portfolio's wealth evolves from period t to t+1 as a function of the current portfolio's return r_{P_t} and next period's cash flow through:

$$W_{t+1} = W_t\left(1+r_{P_t}\right) + C_{t+1}$$

Multi-period portfolio construction seeks to maximize the expected terminal wealth $E(W_T|W_0, C)$ subject to pre-specified cash flows while penalizing for the variance of terminal wealth $V(W_T|W_0, C)$. Mathematically, we solve the following optimization problem:

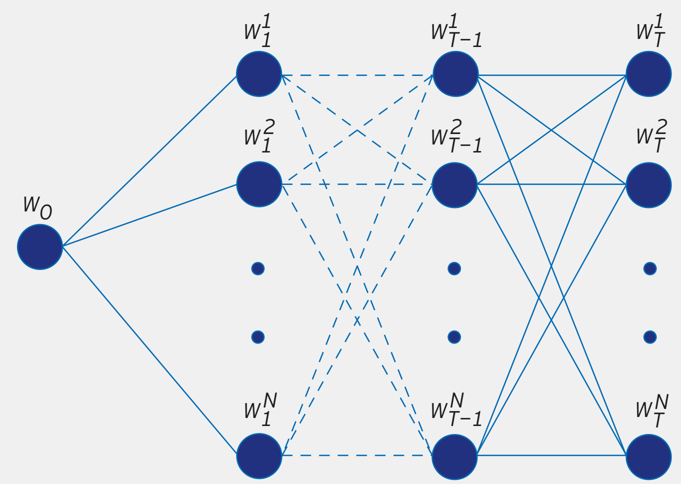
$$\underset{P = P_0, P_1, \ldots, P_{T-1}}{\text{maximize}} \; E(W_T \; W_0, C) - \lambda V(W_T|W_0, C)$$

where $P = P_0, P_1, \ldots, P_{T-1}$ are the portfolios over time. In the equation above, the variance multiplied by a risk aversion parameter λ. As an example, an investor who is not sensitive to risk will choose $\lambda = 0$ and will effectively maximize expected wealth. On the opposite end of the spectrum, an investor who only cares about risk will choose a very high value of λ and will effectively minimize the variance of the terminal wealth. By varying λ in the above formulation we are able to create a full efficient frontier.

We conduct the multi-period optimal portfolio construction exercise in the presence of cash flows by following the stochastic dynamic programming (SDP) principle. This means, given a financial goal of interest such as maximizing the probability distribution's mean value and minimizing its variance, we start the optimization process at the end of the investment horizon at time $t = T\text{-}1$ and work backward in time to $t = 0$. Along the way we solve a series of portfolio optimization problems that produce the optimal weights needed to a) achieve the optimal future outcome and b) use these optimal weights to solve the problem in the previous time period.

More specifically, starting at time $t = T\text{-}1$, we create a grid of N points of future wealth $W_T^1, \ldots, W_T^N$ as well as a grid of N points of current wealth $W_{T-1}^1, \ldots, W_{T-1}^N$. We perform a portfolio optimization process for each of the current nodes in $W_{T-1}^1, \ldots, W_{T-1}^N$ and collect the optimal portfolio weights in the corresponding nodes. Once we have solved the problem at all nodes at time $t = T\text{-}1$, we lock down the optimal weights at the $t = T\text{-}1$ wealth nodes, namely, any portfolio strategy at $t = T\text{-}2$ will use the $t = T\text{-}1$ portfolio weights to reach at $t = T$ portfolio wealth. Once this is determined, we solve the same optimization problem for all the nodes at $t = T\text{-}2$ and continue working backwards until we reach $t = 0$ where we are just solving the optimization problem at only one node, i.e., the starting portfolio wealth W_0. Graphically, we represent the SDP strategy by the tree in Figure 34. We note that at each future point in time we can obtain the average of the optimal weights across the nodes to give us the average glidepath.

Figure 34: The dynamic programming optimization tree



Source: Invesco.

Investment risks

The value of investments and any income will fluctuate (this may partly be the result of exchange rate fluctuations) and investors may not get back the full amount invested.

Diversification and asset allocation do not guarantee a profit or eliminate the risk of loss.

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